Quasicrystals, bi-Lipschitz Equivalence and Bounded Movement.

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Separated net

Definition

A subset A of metric space M is called *separated net* if for some constants r and R following condition holds:

- for every two points $x_1, x_2 \in A$ distance $\boldsymbol{d}_M(x_1, x_2) \geq r$ and
- for every point $y \in M$ $\boldsymbol{d}_M(y, A) \leq R$.

Bi-Lipschitz equivalence

Definition

Two sets $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ in two possibly different metric spaces are called *bi-Lipschitz equivalent* if there exist a bijection $f : A_1 \longrightarrow A_2$ and a constant $L \geq 1$ such that for any two points $x, y \in A_1$ the following inequality holds:

$$\frac{1}{L} \cdot \boldsymbol{d}_{A_1}(x, y) \leq \boldsymbol{d}_{A_2}(f(x), f(y)) \leq L \cdot \boldsymbol{d}_{A_1}(x, y).$$

Two bi-Lipschitz equivalent sets we will designate $A_1 \underset{\mathrm{Lip}}{\sim} A_2$.

Examples

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- Also any set is equivalent to its affine image.

The problem of Gromov and Furstenberg

Problem

Find a practical criterion on a metric space M that would insure a bi-Lipschitz equivalence between every two separated nets in M.

Some non-Euclidean results

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- P. Papasoglu, 1995. Any two homogeneous trees of valence at least 3 are bi-Lipschitz equivalent.
- O. Bogopolskii, 1997. Any two separated nets in hyperbolic space $\mathbb{H}^d$ are equivalent.

Euclidean results

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Euclidean results

- It is evident that in $\mathbb{R}^1$ any two separated nets are equivalent.
- D. Burago, B. Kleiner and in the same time C. McMullen, 1998. There exist a separated net in $\mathbb{R}^d$, $d > 1$ which is **not** equivalent to lattice.

Bounded distance sets

Lemma

Assume that for two separated nets A and B there exist a bijection $f : A \longrightarrow B$ and a constant $\zeta > 0$ such that for every point $x \in A$ the inequality $\mathbf{d}(x, f(x)) \leq \zeta$ holds. Then $A \underset{\text{Lip}}{\sim} B$.

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For $\mathbb{H}^d$ O.Bogopolskii proved not only that any two Delone sets are bi-Lipschitz equivalent but they are also at bounded distance.

Quasicrystals (model sets)

Definition

Let Γ be a lattice in $\mathbb{R}^n \times \mathbb{R}^m$, $\pi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be projections, such that $\pi_1|_\Gamma$ is injective, and $\pi_2(\Gamma)$ is dense in $\mathbb{R}^m$. Let $W \subset \mathbb{R}^m$ be a compact set — the *window* — such that the closure of the interior of W equals W.
This is summarized in the following diagram, which is called *cut-and-project scheme*.

$$\begin{array}{ccccc} \mathbb{R}^n & \xleftarrow{\pi_1} & \mathbb{R}^n \times \mathbb{R}^m & \xrightarrow{\pi_2} & \mathbb{R}^m \\ \cup & & \cup & & \cup \\ V & & \Gamma & & W \end{array}$$

Definition

Then

$$V := V(\mathbb{R}^n, \mathbb{R}^m, \Gamma, W) = \{\pi_1(x) \,|\, x \in \Gamma,\ \pi_2(x) \in W\}$$

is called a *canonical model set*.
The space $\mathbb{R}^n$ is called *physical space* and $\mathbb{R}^m$ is called *internal space*
If $\mu(\partial(W)) = 0$, then V is called *regular* model set.
If $\partial(W) \cap \pi_2(\Gamma) = \varnothing$, then V is called *generic* model set.
Also we can can consider a case with open bounded window W.

This definition can be generalized for any locally compact Abelian groups.

Example of model set

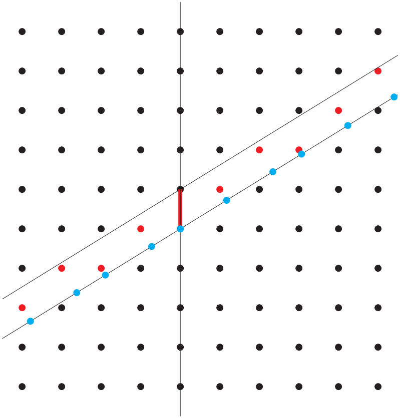

Figure: Fibonacci quasilattice $V(\mathbb{R}^1, \mathbb{R}^1, \mathbb{Z}^2, [0,1))$ with physical space $\mathbb{R}^1 : y = \frac{\sqrt{5}+1}{2}x$.

Example of model set II

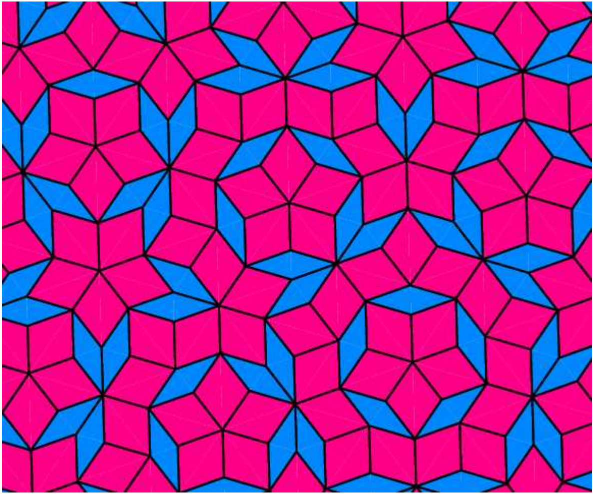

Figure: Penrose tiling.

Quasicrystals and bounded movement

Theorem (M.Duneau, C.Oguey, 1990.)

If window W is a translation copy of a fundamental domain of π_2-projection of some m-sublattice of Γ then the correspondent regular model set $V(\mathbb{R}^n, \mathbb{R}^m, \Gamma, W)$ is at bounded distance from $\mathbb{Z}^n$.

Lemmas of Duneau and Oguey.

Lemma

If Λ_1 *and* Λ_2 *are two lattices in* $\mathbb{R}^d$ *of the same density then they are at the bounded distance.*

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Lemma

If Λ_1 and Λ_2 are two non-intersecting lattices in $\mathbb{R}^d$ of densities ρ_1 and ρ_2 respectively then $\Lambda_1 \cup \Lambda_2$ is at bounded distance from any lattice with density $\rho_1 + \rho_2$.

Constructing more windows

This theorem can be used to obtain quasicrystals with more complicated windows which are at bounded distance from a lattice.

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Consider a quasicrystal $V(\mathbb{R}^2, \mathbb{R}^2, \mathbb{Z}^4, W)$ where W is a regular octagon and basic vectors of $\mathbb{Z}^4$ projected on halves of diagonals of the octagon.

Octagon as a union of fundamental domains

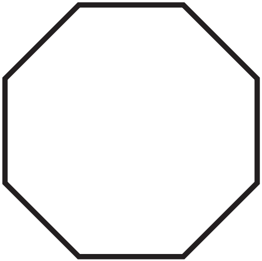

Figure: Octagon can not be a fundamental domain of any 2-lattice.

Octagon as a union of fundamental domains

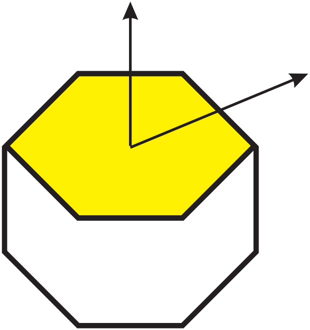

Figure: Hexagon is a fundamental domain of a sublattice.

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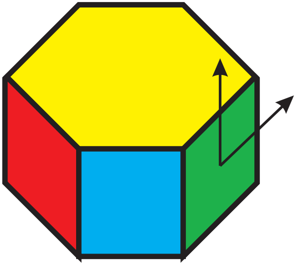

Figure: Each of three parallelograms is a fundamental domain.

Quasicrystals and bounded movement II

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Theorem (H.Kesten, 1966)

The only quasicrystals $V(\mathbb{R}^1, \mathbb{R}^1, \Gamma, W)$ *that satisfy the condition of Duneau and Oguey (possibly with operations of union and subtraction of windows as a sets) are at bounded distance from some one-dimensional lattice.*

Quasicrystals and bi-Lipschitz equivalence

- Consider a real space $\mathbb{R}^3$ and a model set $V = V(\mathbb{R}^2, \mathbb{R}^1, \mathbb{Z}^3, W)$ where the physical space $\mathbb{R}^2$ is a plane $\pi : z = \alpha x + \beta y$.

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Theorem (D. Burago, B. Kleiner, 2002.)

If α satisfies a condition $\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^d}$ for some constants $C > 0$ and $d > 2$ and every natural p, q then the set V is bi-Lipschitz equivalent to the lattice $\mathbb{Z}^2$.

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Theorem (Y.Solomon, 2007.)

Delone set created from the centers of Penrose tiling is bi-Lipschitz equivalent to a lattice.

Quasicrystals and bi-Lipschitz equivalence

Theorem (D.Frettlöh, A.Garber, 2009.)

Any two-dimensional regular generic model set is bi-Lipschitz equivalent to a lattice.