Toral rank conjecture for moment-angle complexes

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Abstract

In this paper we introduce an operation on the set of simplicial complexes, which we shall call "doubling operation". We show that the moment-angle complex $\mathcal{Z}_K$ is the real moment-angle complex $\mathbb{R}\mathcal{Z}_{L(K)}$ for simplicial complex $L(K)$ obtained from K by applying "doubling operation". As an application of this operation we prove the toral rank conjecture for $\mathcal{Z}_K$ by estimating the lower bound of the cohomology rank (with rational coefficients) of the real moment-angle complexes $\mathbb{R}\mathcal{Z}_K$. In, [4] the "doubling operation" for the polytopes was defined and the same result was proved for the class of the moment-angle manifolds, so this article can be considered as the extension of the previous one.

1 Doubling operations

Here we give the definition of the "doubling operation" and discuss its main properties.

Definition 1.1. Let K be an arbitrary simplicial complex on the vertex set $[m] = \{v_1, \dots, v_m\}$. The *double* of K is the simplicial complex $L(K)$ on the vertex set $[2m] = \{v_1, v_1', \dots, v_m, v_m'\}$ determined by the following condition: $\omega \subset [2m]$ is the minimal (by inclusion) missing simplex of $L(K)$ iff ω is of the form $\{v_{i_1}, v_{i_1}', \dots, v_{i_k}, v_{i_k}'\}$, where $\{v_{i_1}, \dots, v_{i_k}\}$ is a missing simplex of K.

If $K = \partial P^*$ is a boundary of the dual of the simple polytope P, then $L(K)$ coincides with $L(P)^*$, see the definition 1 in [4].

Examples.

- If $K = \Delta^m$ is the $(m-1)$-dimensional simplex, then $L(K) = \Delta^{2m}$.
- If $K = \partial\Delta^m$ is the boundary of the $(m-1)$-dimensional simplex, then $L(K) = \partial\Delta^{2m}$.

It is easy to see that "doubling operation" respects join of the simplicial complexes i.e. $L(K_1 * K_2) = L(K_1) * L(K_2)$.

Given a simplicial complex K we denote by $\operatorname{mdim} K$ the minimal dimension of the maximal by inclusion simplices. Thus, for any K $\operatorname{mdim} K \leqslant \dim K$, and K is pure iff $\operatorname{mdim} K = \dim K$.

The following lemma is the direct corollary from the definitions.

Lemma 1.2. *Let K be a simplicial complex on $[m]$, then* $\dim L(K) = m + \dim K$ *and* $\operatorname{mdim} L(K) = m + \operatorname{mdim} K$.

2 K-powers

Definition 2.1. Let (X, A) be a pair of CW — complexes. For a subset $\omega \subset [m]$ we define

$$(X, A)^\omega := \{(x_1, \dots, x_m) \in X^m | x_i \in A \text{ for } i \notin \omega\}.$$

Now let K be a simplicial complex on $[m]$. The K-*power* of the pair (X, A) is

$$(X, A)^K := \bigcup_{\omega \in K} (X, A)^\omega.$$

In this paper we shall consider two examples of K-powers (see [1]):

- Moment-angle complexes $\mathcal{Z}_K = (D^2, S^1)^K$.
- Real moment-angle complexes $\mathbb{R}\mathcal{Z}_K = (I^1, S^0)^K$.

The next lemma explains the usefulness of the notion of "doubling operation" in the studying of the relationship between moment-angle complexes and real moment-angle complexes.

Lemma 2.2. *Let (X, A) be a pair of CW — complexes and K be a simplicial complex on the vertex set $[m]$. Consider a pair $(Y, B) = (X \times X, (X \times A) \cup (A \times X))$. For this pair we have:*

$$(Y, B)^K = (X, A)^{L(K)}.$$

In particular $\mathcal{Z}_K = \mathbb{R}\mathcal{Z}_{L(K)}$.

Proof. For a point $\mathbf{y} = (y_1, \dots, y_m) \in Y^m$ we set

$$\omega_Y(\mathbf{y}) = \{v_i \in [m] \,|y_i \in Y \backslash B\} \subset [m].$$

For a point $\mathbf{x} = (x_1, x_1', \dots, x_m, x_m') \in X^{2m}$ the subset $\omega_X(\mathbf{x}) \subset [2m]$ is defined in a similar way. Let $\mathbf{y} = (y_1, \dots, y_m) = ((x_1, x_1'), \dots, (x_m, x_m')) \in Y^m = X^{2m}$. It follows from the definition of the K-powers that $\mathbf{y} \notin (Y, B)^K$ iff $\omega_Y(\mathbf{y}) \notin K$. The latter is equivalent to the condition $\omega_X(\mathbf{x}) \notin L(K)$, where $\mathbf{x} = (x_1, x_1', \dots, x_m, x_m')$, since if $\omega_Y(\mathbf{y}) = \{v_{i_1}, \dots v_{i_k}\}$ then $\omega_X(\mathbf{x}) \supset \{v_{i_1}, v_{i_1}', \dots, v_{i_k}, v_{i_k}'\}$. Therefore

$$\mathbf{y} \notin (Y, B)^K \Leftrightarrow \mathbf{x} \notin (X, A)^{L(K)}$$

and the statement of the lemma is proved. □

Example. Let $K = \partial\Delta^2$ be the boundary of 1-simplex. Then we get decomposition of 3-dimensional sphere:

$$\mathcal{Z}_K = D^2 \times S^1 \cup S^1 \times D^2 = S^3.$$

On the other hand $L(K) = \partial\Delta^4$ and $\mathbb{R}\mathcal{Z}_{L(K)} = \partial I^4 = S^3$ is the boundary of the standard 4-dimensional cube. So, in accordance with the lemma, $\mathcal{Z}_K = \mathbb{R}\mathcal{Z}_{L(K)}$.

3 Toral rank conjecture

Let X be a finite-dimensional topological space. Denote by $\mathrm{trk}(X)$ the largest integer for which X admits an *almost free* $T^{\mathrm{trk}(X)}$ action.

Conjecture (Halperin's toral rank conjecture, [3])**.**

$$\mathrm{hrk}(X, \mathbb{Q}) := \sum \dim H^i(X, \mathbb{Q}) \geqslant 2^{\mathrm{trk}(X)}$$

Moment-angle complexes provide a big class of spaces with torus action, since there is natural coordinatewise T^m action on the space $\mathcal{Z}_K$. In fact for some r one can choose subtorus $T^r \subset T^m$ such that the action $T^r \colon \mathcal{Z}_K$ is almost free. Our aim is to estimate the maximal rank of such subtorus and the lower bound of $\mathrm{hrk}(\mathcal{Z}_K, \mathbb{Q})$.

Lemma 3.1. *Let K be $(n-1)$-dimensional simplicial complex on the vertex set $[m]$. Then the rank of subtorus $T^r \subset T^m$ that acts almost freely on $\mathcal{Z}_K$ is less or equal to $m - n$.*

Proof. For a subset $\omega \subset [m]$ we set $T^\omega = (T, e)^\omega$ (see definition of K-powers), where $e \in T$ is identity. It is easy to see that isotropy subgroups of the action $T^m : \mathcal{Z}_K$ are of the form T^ω, $\omega \in K$. Therefore $T^r \subset T^m$ acts almost freely iff the set $T^r \cap T^\omega$ is finite for any $\omega \in K$.

Let σ be the simplex of the dimension $(n-1)$. Since the intersection $T^r \cap T^\sigma$ of two subtori in T^m is finite,

$$\operatorname{rk} T^r + \operatorname{rk} T^\sigma \leqslant \operatorname{rk} T^m,$$

thus $r \leqslant m - n$. □

Remark. In fact for any $(n-1)$-dimensional complex K there is subtorus $T^r \subset T^m$ of the rank $r = m - n$ that acts on $\mathcal{Z}_K$ almost freely, [2] 7.1.

Now we prove our main result about the cohomology rank of the real moment-angle complexes.

Theorem 3.2. *Let K be a simplicial complex on the vertex set $[m]$ with* $\operatorname{mdim} K = n - 1$. *Then*

$$\operatorname{hrk}(\mathbb{R}\mathcal{Z}_K, \mathbb{Q}) \geqslant 2^{m-n}.$$

We first formulate one general lemma.

Lemma 3.3. *Let (X, A) be a pair of CW-complexes; let $U(A)$ be a neighbourhood of A in X such that $(U(A), A) \simeq (A \times [0; 1), A \times \{0\})$. Then the cohomology rank of the space Y, obtained from the two copies of X by gluing them together along A, $Y = X_1 \bigcup_A X_2$, satisfies inequality:*

$$\operatorname{hrk}(Y, \mathbb{Q}) \geqslant \operatorname{hrk}(X, \mathbb{Q}).$$

This fact is direct consequence of the Mayer Vietoris long exact sequence.

Proof of the theorem. We shall prove this fact by induction on m. The base of induction is trivial.

Assume this statement is true for the complexes with less than m vertices and K is the complex with m vertices.

The real moment-angle complex is a subspace of the m-dimensional cube $\mathbb{R}\mathcal{Z}_K \subset [-1; 1]^m$. Denote by $(x_1, \dots, x_m)$ coordinates in $[-1; 1]^m$. Assume that the vertex v_1 belong to the maximal (by inclusion) simplex of K of the dimension $\operatorname{mdim} K = n - 1$. Consider the decomposition of $\mathbb{R}\mathcal{Z}_K = M_+ \cup_X M_-$, where

$M_+ = \{\vec{x} \in \mathbb{R}\mathcal{Z}_K \subset \mathbb{R}^m \mid x_1 \geqslant 0\}$,
$M_- = \{\vec{x} \in \mathbb{R}\mathcal{Z}_K \subset \mathbb{R}^m \mid x_1 \leqslant 0\}$,
$X = \{\vec{x} \in \mathbb{R}\mathcal{Z}_K \subset \mathbb{R}^m \mid x_1 = 0\}$.

It is easy to see that the pair (M_+, X) satisfies the hypothesis of the lemma 3.3, so

$$\operatorname{hrk}(\mathbb{R}\mathcal{Z}_K, \mathbb{Q}) \geqslant \operatorname{hrk}(X, \mathbb{Q}).$$

Now lets describe the space X more explicitly. Let k be the number of vertices in the complex $\operatorname{lk} v_1$. Then X is just the disjoint union of the 2^{m-k-1} copies of the space $\mathbb{R}\mathcal{Z}_{\operatorname{lk} v_1}$. Moreover, since v_1 is vertex of the maximal (by inclusion) simplex of the minimal dimension n, so $\operatorname{mdim} \operatorname{lk} v_1 = n - 2$. Thus, by the hypothesis of induction

$$\operatorname{hrk}(X, \mathbb{Q}) = 2^{m-k-1} \operatorname{hrk}(\mathbb{R}\mathcal{Z}_{\operatorname{lk} v_1}, \mathbb{Q}) \geqslant 2^{m-k-1} \cdot 2^{k-(n-1)} = 2^{m-n}$$

The step of induction is proved. □

Now let's turn our attention to the moment-angle complexes. Combining the results of lemma 1.2, lemma 2.2 and theorem 3.2 we have:

$$\operatorname{hrk}(\mathcal{Z}_K, \mathbb{Q}) = \operatorname{hrk}(\mathbb{R}\mathcal{Z}_{L(K)}) \geqslant 2^{2m - \operatorname{mdim} L(K) - 1} = 2^{m - \operatorname{mdim} K - 1} \geqslant 2^{m - \dim K - 1}.$$

Thus the *toral rank conjecture* holds for the action of subtori of T^m on the moment-angle complexes $\mathcal{Z}_K$.

The cohomology ring of $\mathcal{Z}_K$ was calculated in [1]. One of the corollaries of this computation and Hochster's theorem states (see [1], theorem 8.7):

Theorem 3.4.
$$H^*(\mathcal{Z}_K,\mathbb{Z}) \cong \bigoplus_{\omega\subset[m],\ p\geqslant -1} \tilde{H}^p(K_\omega,\mathbb{Z}),$$
where K_ω *is the restriction of* K *on the subset* $\omega\subset[m]$.

In view of this theorem we can reformulate our main result as follows:
$$\dim \bigoplus_{\omega\subset[m]} \tilde{H}^*(K_\omega,\mathbb{Q}) \geqslant 2^{m-n},$$
for any simplicial complex K on $[m]$ with $\operatorname{mdim} K = n-1$.

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