AN INDEX OF AN EQUIVARIANT VECTOR FIELD AND ADDITION THEOREMS FOR PONTRJAGIN CHARACTERISTIC CLASSES

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Abstract.

The theory of indices of Morse–Bott vector fields on a manifold is constructed and the famous localization problem for the transfer map is solved on its base in the present paper. As a consequence, we obtained addition theorems for the universal Pontrjagin characteristic classes in cobordisms. These results gave us a possibility to complete the construction, which was begun more than twenty years ago, of the universal characteristic classes' theory.

References: 20 items.

§ 1. Introduction

The transfer map construction for fibre bundles [2], [3] is one of the most fundamental notion in current algebraic topology. This construction makes it possible to vitally develop and extend applications of the direct image method connected with such notions as the direct image for representations, a trace of an algebraic extension, the transfer map for vector bundles over covering, the Gysin map etc. (see [1]–[6], [10]–[12], [17]).

The theory of indices of Morse–Bott vector fields on a manifold is constructed and the famous localization problem for the transfer map is solved on its base in the present paper. This result is remote generalization of the classical Poincare–Hopf theorem expressing the Euler characteristic of a manifold as the sum of zero indices of a regular tangent vector field.

As a consequence, we obtained the formulae expressing the universal (that is, with its values in cobordism theories) Pontrjagin characteristic classes of the sum of two vector bundles in terms of summands' characteristic classes (the addition theorems). These results gave us a possibility to complete the construction, which

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was begun more than twenty years ago [6], of the universal characteristic classes' theory.

The contents of the article is as follows. In § 2 we recall the transfer map construction for fibre bundles and the related definitions of the fiber bundle index. In § 3 we introduce the index $Ind_s(F_l)$ of a Morse–Bott vector field s on a smooth manifold F in a neighborhood of its zero submanifold F_l. This index takes its values in the cohomotopy group $\pi^0_S(F_l^+)$. We give an example of the Morse–Bott vector field on the projective plane. In a neighborhood of the projective line the index of this field equals to $-1+u$, where u is the generator of the group $\pi_1^S = Z_2$. We prove important Theorem 3.1 of the transfer map localization for a smooth manifold (as a fibre bundle over a point) using the Morse–Bott vector field on the manifold. As a consequence, we obtain an expression for the Euler characteristic $\chi(F)$ in terms of the Euler characteristic $\chi(F_l)$ and indices $Ind_s(F_l)$ for the Morse–Bott vector field s with the collection of the connected zero submanifolds $\{F_l\}$ (Theorem 3.2). In § 4 we prove in general case the localization theorem for the transfer map of an arbitrary fiber bundle with a smooth fiber. We demonstrate that the known results [5], [11] in this direction can be reduced to special cases of the theorem.

The rest part of the work is devoted to the applications of the results obtained to the theory of Pontrjagin characteristic classes of real vector bundles. In § 5 we construct an important Morse–Bott vector field over grassmannization of the Whitney sum of two real vector bundles. We describe connected components of the zero submanifolds and calculate the indices of this field in their neighborhoods.

In § 6 we obtain one of the central results of the article: the addition theorem for the universal Pontrjagin classes of real vector bundles. In details we investigate the case of the first half–integer class $p_{1/2}$. We claim that on the image of the map

$$p_{1/2}\colon KO^*(X) \to U^2(X)$$

there is a new algebraic operation that cannot be reduced to any formal group. Call attention to the fact that in our short paper [9] we made an inaccuracy in the formula for $p_{1/2}(\xi \oplus \zeta)$ in complex cobordisms, which we correct in the present work.

In the last section we investigate Pontrjagin characteristic classes of stable complex vector bundles. We prove that if a real vector bundle admits a stable complex structure, its half–integer Pontrjagin classes are equal to zero while its integer Pontrjagin classes have a realization in symplectic cobordisms.

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§ 2. The Index of a Smooth Fibering

In this section we shall give definitions of the main notions connected with the transfer map of Becker–Gottlieb [3].

Let us consider two topological spaces X and Y with marked points and the set of homotopic classes $[X, Y]$ of all maps from X to Y, sending a marked point to

another marked one. We denote by $\{X,Y\}$ the group of stable homotopic classes of maps
$$\{X,Y\} = \lim_{n\to\infty}[S^n\wedge X, S^n\wedge Y].$$

Definition 2.1. *A zero–dimensional cohomotopy group of a space X with a marked point* is the group
$$\pi^0_S(X) = \{X,S^0\} = \lim_{n\to\infty}[S^n\wedge X, S^n].$$

Let X^+ denote a disjoint union of the space X and a point, which we shall assume to be a marked point. If the marked point in the space X is already given, the space $S^n\wedge X^+$ is homotopy–equivalent to $S^nX\vee S^n$, and therefore,
$$\pi^0_S(X^+) = \pi^0_S(X)\oplus Z.$$
We shall denote by ϵ the canonical projection $\pi^0_S(X^+)\to Z$. If the space X is the sphere S^k,
$$\mathrm{Ker}(\epsilon)\cong\pi^S_k$$
where π^S_k is the k-th stable homotopy group of spheres.

Every fibering (E,F,B,p) with the smooth compact closed fiber F defines the canonical element in the group $\{B^+,E^+\}$. Existence of this element is provided by the transfer map of Becker and Gottleib [3]. Let us give the construction of this element in case of a smooth fibering E with the compact base (for arbitrary cell bases see the definition in [8]).

The space of the fibering E can be embedded into $\mathbb{R}^n$ for suitable n. Let i denote this embedding. We consider the space $\tau_F(E)$ of tangents along the fiber of the bundle E and the normal bundle $\nu(E)$ of the inclusion $p\times i\colon E\subset B\times\mathbb{R}^n$. It is easy to check that $\tau_F(E)\oplus\nu(E)$ is a trivial n-dimensional vector bundle over E [3]. Let $T(\xi)$ denote the Thom space of an arbitrary vector bundle ξ.

Definition 2.2. *The transfer map of a smooth fiber bundle* (E,F,B,p) is the composition of the maps
$$\tau(p)\colon S^n\wedge B^+\to T\nu(E)\to T\big(\nu(E)\oplus\tau_F(E)\big) = T[\bar n]$$
where the first map is the Pontrjagin–Thom map representing the glueing of the tubular neighborhood's exterior of the submanifold E in the manifold $B\times\mathbb{R}^n$, while the second map is the inclusion of the fiber bundle $\nu(E)$ on the direct summand.

Notice that the Thom space of the trivial n-dimensional vector bundle over B is the n-fold suspension over B^+. Therefore, the transfer map defines a certain element of the group $[S^n\wedge B^+, S^n\wedge E^+]$. The image of this element in the group $\{B^+,E^+\}$ is independent of a choice of the embedding [3]. Thus, the transfer map correctly defines an element, which we shall denote by $\{\tau(p)\}$, of the group $\{B^+,E^+\}$.

For any cohomology theory $h^*(\cdot)$ a stable map from X to Y uniquely defines groups' homomorphism

$$h^*(Y) \to h^*(X).$$

We shall denote by $\tau(p)^*$ the corresponding homomorphism for the transfer map of the fibering (E, F, B, p). Let π represent the projection on the first summand $\pi : S^n \wedge E^+ \to S^n$.

Definition 2.3. *An index $I(E)$ of the fiber bundle $p : E \to B$ is an element* $\{\pi \circ \tau(p)\} \in \pi^0_S(B^+)$. An index of this fiber bundle in the cohomology theory $h^*(\cdot)$ is the element $I_h(E) = \tau(p)^*(1) \in h^0(B^+)$ (we consider the subgroup h^0 in the group h^* as a ring with unit). Thus, $I(E)$ is the fiber bundle's index in the stable cohomotopy theory $\pi^*_S(\cdot)$.

Notice that for any cohomology theory $h^*(\cdot)$ with unit there is a canonical transformation of cohomology theories (see [19])

$$\mu_h : \pi^*_S(\cdot) \to h^*(\cdot)$$

which transforms the unit of the theory $\pi^*_S(\cdot)$ into the unit of the theory $h^*(\cdot)$. In the theory $h^*(\cdot)$ the index of the fiber bundle $p : E \to B$ is equal to $\mu_h\big(I(E)\big)$.

The indices of the fiber bundles are already non–trivial in the simplest case for the fiberings over a point. In that case $\pi^0_S(B^+) \cong Z$, and the index of the fibering is equal to the Euler characteristic of the fiber [3].

§ 3. Localization

The index of the tangent vector field on the smooth manifold in a neighborhood of a nondegenerate zero is the fundamental characteristic of the zero considered [14]. This index is defined as determinant's sign of the linear part of the vector field in a small neighborhood of the zero [14]. In this section we claim that in a neighborhood of the degenerate zero the differential of the field gives an important characteristic of this zero. In terms of the characteristics introduced one can express the coefficients, for which the transfer map of the smooth manifold can be represented by the transfer maps' linear combination of the zero submanifolds of the vector field on the initial manifold.

Before proving the main theorem we shall give an example that will help one to understand the transfer map localization property. Let us begin with necessary notation.

Each element of the group $\pi^0_S(B^+)$ can be determined by a continuous map as $S^n \wedge B^+ \to S^n$ for n large enough. Let us denote by π the projection on the first factor in the space $S^n \wedge B^+$. The representative γ of an element of the group $\pi^0_S(B^+)$ defines a map

$$f_\gamma \colon B \to \mathrm{Map}(S^n, S^n)$$

by the formula $f_\gamma(b)(y) = \gamma(y, b)$. The inverse is also true. Any continuous map $f : B \to Map(S^n, S^n)$ defines a certain element of $\pi^0_S(B^+)$ by the formula $\gamma_f = \{\pi\big(f(b)(y), b\big)\}$. Let us identify S^n with $\mathbb{R}^n \cup \{\infty\}$ and define the action $\mathrm{GL}(n, \mathbb{R})$

on S^n by this identification. Even in case of $f(b) \in \mathrm{GL}(n, \mathbb{R})$ for any $b \in B$ the correspondent element of the cohomotopy group can take nonzero value after the projection onto $\mathrm{Ker}(\epsilon)$.

Example 3.1. Let $B = S^1$ and ξ_1 denote the nontrivial one–dimensional real vector bundle over S^1. We have $\xi_1 \oplus \xi_1$ is the trivial two-dimensional real vector bundle. Let us consider the map

$$S^2 \wedge S^{1+} = T(\xi_1 \oplus \xi_1) \xrightarrow{(1,-1)} T(\xi_1 \oplus \xi_1) = S^2 \wedge S^{1+} \xrightarrow{\pi} S^2. \tag{3.1}$$

Since $S^2 \wedge S^{1+} \sim S^3 \vee S^2$, the map constructed above defines two maps $\alpha_1 : S^3 \to S^2$ and $\alpha_2 : S^2 \to S^2$. It is easy to check that α_2 is the reflection relatively to the central plane. Let us show that α_1 defines a generator of the group $\pi_3(S^2)$. Identify S^2 and $\mathbb{C}^1 \cup \{\infty\}$ setting the infinite point to be a marked one. We consider the preimage of the zero under the map α_1. It is the zero section in the bundle $\xi_1 \oplus \xi_1$, that is, the circle. The whole preimage $\mathbb{C}^1 \subset S^2$ under the map α_1 is the space of the vector bundle $\xi_1 \oplus \xi_1$. The splitting of the trivial two-dimensional real vector bundle over S^1 in the direct sum $\xi_1 \oplus \xi_1$ defines the chosen basis in each fiber of the bundle

$$\big\langle e_1(\phi), e_2(\phi)\big\rangle = \langle e^{i\phi/2}, ie^{i\phi/2}\rangle$$

where $\phi \in [0, 2\pi]$ is the angular coordinate on the zero section (that is, on S^1). For any point $\{\phi, x + iy\}$ from $S^1 \times \mathbb{C}^1 \subset S^3$ its image under the map α_1 is

$$\alpha_1\big(\{\phi, x + iy\}\big) = \big\{(x - iy)e^{i\phi}\big\} \in C^1.$$

In coordinates $(z_1, z_2) \in \mathbb{C}^1 \times \mathbb{C}^1$, $|z_1| = 1$, the map α_1 can be expressed by the formula

$$\alpha_1(z_1, z_2) = \bar{z}_2 z_1,$$

which shows that the map α_1 is homotopic to the projection in the Hopf bundle.

Therefore, the element of the group $\pi^0_S(S^{1+})$ corresponding to the map (3.1) is $-1 + u$, where u is the generator of the group $\pi^S_1 \cong Z_2$.

Now we formulate the localization property, which at first will be proved for smooth manifolds, that is, in case of fiber bundles over a point.

We shall use the notion of a tangent vector field of the Morse–Bott type (see [16]); Morse vector fields are the special case of such fields.

Definition 3.1. A tangent vector field on a smooth manifold F is called *a Morse–Bott field* in case of the two following conditions are satisfied:

1) the zero set of the vector field forms a finite set of connected compact closed submanifolds in F;

2) the kernel of the Jacobian matrix of the vector field at the points of the zero submanifold coincides with the tangent space of the zero submanifold.

Let us consider the connected component F_l of the zero set of the Morse–Bott vector field. Let ν_l be the normal bundle of the embedding $F_l \subset F$. The bundle space ν_l can be identified with the tubular neighborhood of the submanifold F_l in F using the exponential map [14]. The restriction of the Morse–Bott vector field on the tubular neighborhood defines the vector field on the space ν_l. Owing to the condition 2) the last vector field must have everywhere non-zero projection on the bundle of tangents along the fibers of the bundle ν_l in a small enough neighborhood of the zero section. Thus, without loss of generality, it can be assumed that the vector field constructed on the space ν_l is tangent along the fibers of the bundle ν_l everywhere. Let us denote by s_l this vector field

$$s_l\colon \nu_l \to \hat{\tau}(\nu_l)$$

where $\hat{\tau}(\nu_l)$ is a bundle of tangents along the fibers. Let p be the projection in the bundle ν_l, then $\hat{\tau}(\nu_l) \cong p^*\nu_l$ (see, for example, [13]). Thus, the vector field s_l allows us to construct the map $\psi_l^s\colon \nu_l \to \nu_l$ by the rule

$$\psi_l^s(v) = \hat{p}s(v)$$

where $\hat{p}$ is the bundle map $\hat{p}\colon p^*\nu_l \to \nu_l$.

Let us consider a smooth compact closed manifold F and an arbitrary Morse–Bott vector field s on F. Let $F_1, \ldots, F_m$ be the whole collection of connected compact closed zero submanifolds of the vector field s on F. Let us fix the embedding of the manifold F in the n-dimensional Euclidean space:

$$F_1, \ldots, F_m \subset F \subset \mathbb{R}^n.$$

Using the vector field s we correspond the map

$$j_l\colon S^n \wedge F_l^+ \to S^n \wedge F_l^+$$

to each submanifold F_l as follows. The embedding of the smooth manifolds $F_l \subset F \subset \mathbb{R}^n$ induces the splitting of the trivial n-dimensional vector bundle over F_l into the direct sum of the tangent bundle τ_l of F_l, the normal bundle ν_l of the embedding $F_l \subset F$, and the restriction on F_l of the normal bundle ν of the embedding $F \subset \mathbb{R}^n$. The field s determines the map $\phi_l^s\colon \nu_l \to \nu_l$. Let us define the map j_l on the subbundles ν and τ_l by identical, nd that on ν_l by ϕ_l^s.

By the above, let π be the projection $S^n \wedge F_l^+ \to S^n$ on the first factor.

Definition 3.2. An element $\{\pi \circ j_l\} \in \pi_S^0(F_l^+)$ will be called *an index* $\mathrm{Ind}_s(F_l)$ *of the vector field* s in a neighborhood of the zero submanifold F_l.

Remark 3.1. An index of the isolated zero of the vector field coincides with $\epsilon\big(\mathrm{Ind}_s(pt)\big)$ and is equal to the ordinary index of the isolated zero [14].

Example 3.2. Using Example 3.1 we claim that an index $Ind_s(F_l) \in \pi_S^0(F_l^+)$ can have non–zero component in $\mathrm{Ker}(\epsilon)$.

Let us define an action of the positive real numbers group $\mathbb{R}_+$ on RP^2 by the formula

$$t \circ (x_1 : x_2 : x_3) = (x_1 : x_2 : t^{-1}x_3).$$

The differential of this action determines the tangent vector field s on RP^2 if $t = 1$. All fixed points of the action are zeros of s, that is,

$$RP^1 = \{(x_1 : x_2 : 0)\} \subset RP^2,$$

and the point $(0 : 0 : 1) \in RP^2$. The normal bundle of the embedding $RP^1 \subset RP^2$ can be identified with the nontrivial one-dimensional real vector bundle ξ_1 over S^1. In a neighborhood $RP^1 = S^1 \subset RP^2$ the vector field s is determined by the formula $s(v) = -v$, $v \in \xi_1 \approx N(S^1) \subset RP^2$. Owing to Example 3.1 an index $\mathrm{Ind}_s(RP^1)$ is equal to $-1 + u$, where u is the generator of the group $\pi_1^S \cong Z_2$.

Let the map τ_l be the transfer for the manifold F_l. Let us denote by i the Thom space embedding of the trivial n-dimensional vector bundle over $F_1 \cup \cdots \cup F_m$ in the Thom space of the trivial n-dimensional vector bundle over F that corresponds to the embedding

$$F_1 \cup \cdots \cup F_m \subset F.$$

Theorem 3.1 (the localization property). *The transfer map τ for the manifold F is homotopic to the composition*

$$i \circ (j_1 \circ \tau_1 \vee \cdots \vee j_m \circ \tau_m) \circ g$$

where g is the map sending the sphere S^n into the union of m spheres.

Proof. The embedding of the Thom space

$$\alpha\colon T\nu(F) \to T\big(\nu(F) \oplus \tau(F)\big)$$

from the definition of the transfer map can be changed for a homotopic one using the tangent field. In fact, let us consider a one–parametric family of maps

$$\alpha_t(v) = \alpha(v) \oplus ts\big(\pi(v)\big), \qquad v \in \nu(F),$$

where π is the projection in the bundle $\nu(F)$. Choosing a large enough parameter t we shall get the map, homotopic to α, sending small neighborhoods' exteriors of the zeros of the field s at the marked point of the space $T\big(\nu(F) \oplus \tau(F)\big)$. This means the map τ can be passed through the map g and the union of maps sending S^n to $T\nu(F_l)$, $l = 1, \dots, m$.

To describe properties of the map τ in a neighborhood of the zeros of the field s it will be conveniet to change the map α on a homotopic one using the canonical identification of tubular neighborhoods of the submanifolds F_l and normal bundles of embeddings $F_l \subset F$, $l = 1, \dots, m$.

We denote by exp_l the exponential map that defines a diffeomorphism of unit-disk bundle associated with the normal bundle ν_l of the embedding $F_l \subset F$ and the tubular neighborhood $N(F_l) \subset F$ of the submanifold F_l in F, $l = 1, \dots, m$. We define two functions

$$\lambda_l, \mu_l\colon D^{k_l} \to \mathbb{R}^{k_l}$$

for each $l = 1, \dots, m$, where D^s denotes the unit disk of dimension s, $k_l = \operatorname{codim} F_l$, setting

$$\lambda_l(v) = \begin{cases} |v|, & |v| \leqslant \frac{1}{2}, \\ 1 - |v|, & |v| > \frac{1}{2} \end{cases}$$

and

$$\mu_l(v) = \begin{cases} 0, & |v| \leqslant \frac{1}{2}, \\ 2|v| - 1, & |v| > \frac{1}{2}. \end{cases}$$

Let us consider the map

$$\tilde{\alpha}\colon F \to \tau(F),$$

$$\tilde{\alpha}(x) = \begin{cases} \alpha(x), & x \notin N(F_l), \\ \{\exp\bigl(\mu_l(\exp_l^{-1}(x)) \exp_l^{-1}(x)\bigr), \lambda_l(\exp_l^{-1}(x)) \exp_l^{-1}(x)\}, & x \in N(F_l), \end{cases} \quad l = 1, \dots, m.$$

Since λ_l is homotopic to the zero map and $\mu_l(\exp_l^{-1}(x)) \exp_l^{-1}(x)$ is homotopic to the map $\exp^{-1}(x)$, the map $\tilde{\alpha}$ is homotopic to the embedding on the zero section $F \to \tau(F)$.

We remark that

$$\nu(F)\big|_{N(F_l)} \cong \psi_l(1)^* \nu(F)\big|_{F_l}$$

where $\psi_l(t)$ is the retraction of the normal t-tube $N_t(F_l)$ on F_l. Consequently, the map α is homotopic to the map

$$\beta(v) = \begin{cases} v, & \pi(v) \notin N(F_l), \\ \psi_l\bigl(|\pi\alpha(\pi(v))|\bigr)^*(v), & \pi(v) \in N(F_l), \end{cases} \qquad l = 1, \dots, m.$$

Thus, the map constructed is the result of a small embedding perturbation

$$T\nu(F) \to T\bigl(\nu(F) \oplus \tau(F)\bigr)$$

under which the tubular neighborhoods' fibers of the submanifolds F_l transfer into the fibers of the normal bundles of the embeddings $F_l \subset F$ corresponding them under the exponential map.

Let us construct the map

$$\tau(t) = \bigl(\beta(v) \oplus t\tilde{\alpha}_*\bigl(s(\pi(v))\bigr)\bigr)$$

where $v \in T\nu(F)$, $t \in [0; \infty)$, and we turn t to ∞. We obtain the map $\tau(\infty)$ that sends at infinity the points of the bundle $\nu(F)$ lying in the fibers over exterior of the tubular neighborhoods of the submanifolds F_l.

The fiber of the tubular neighborhood of the submanifold F_l in F under $\tau(\infty)$ maps onto the fiber of the subbundle ν_l (see above). One can neglect the first summand in the sum

$$\beta(v) \oplus t\tilde{\alpha}_*\bigl(s(\pi(v))\bigr)$$

for large t. Thus, the restriction of the map $\tau(\infty)$ to the fiber of the tubular neighborhood of the submanifold F_l in F coincides with the map ϕ_l^s (see above). To complete the proof we have to remark that the image of the map $\tau(\infty)$ coincides with the image of the map i.

Remark 3.2. The conditions imposed on the vector fields in the formulation of the localization property can be weakened. It is enough to require the vector field to be homotopic to the Morse–Bott vector field in the class of the vector fields with the fixed zero set.

Remark 3.3. It is easy to check that if the vector field s is homotopic to the outward normal field on the boundary of the tubular neighborhood of the zero submanifold F_l, the map j_l corresponding to that submanifold is homotopic to the identical map [8]. In particular, this situation always holds in case of the vector field defined by a circle action on the manifold [2].

Remark 3.4. Immediately from the proof of Theorem 3.1 it follows that if the tangent vector field on the manifold has no zeros, the transfer map for this manifold is homotopic to the map into the point.

The localization property and Remark 3.1 allow to immediately obtain the following generalization of Poincare–Hopf Theorem (compare [20]).

Theorem 3.2. *Let F denote a smooth compact closed manifold. Let the Morse–Bott vector filed s be given on F. Assume that zeros of the vector field s form a finite collection of connected compact closed submanifolds $F_1, \dots, F_m$ of the manifold F. Then*

$$\chi(F) = \sum_{l=1}^{m} \epsilon\big(\mathrm{Ind}_s(F_l)\big) \cdot \chi(F_l)$$

where $\chi(F)$ is the Euler characteristic of the manifold F.

Proof. We shall use the notation of Theorem 3.1. At first we shall consider an arbitrary tangent vector field on F with isolated nondegenerated zeros. Let us apply the localization property to this field. We obtain

$$I(F) = \{\pi \circ \tau\} = \sum_{k=1}^{N} \{\pi \circ j_k\} = \sum_{k=1}^{N} \epsilon\big(\mathrm{Ind}_s(z_k)\big)$$

where z_k are zeros of the vector field, $k = 1, \dots, N$. Owing to Remark 3.1 this means that $I(F) = \chi(F)$ is the Euler characteristic of the manifold F.

Applying the localization property to the vector field s we obtain

$$\begin{aligned}\chi(F) = I(F) = \{\pi \circ \tau\} &= \sum_{l=1}^{m} \{\pi \circ j_l \circ \tau_l\} = \sum_{l=1}^{m} \tau(p_l)^* \{\pi \circ j_l\} \\ &= \sum_{l=1}^{m} \epsilon\big(\mathrm{Ind}_s(F_l)\big) \cdot I(F_l) = \sum_{l=1}^{m} \epsilon\big(\mathrm{Ind}_s(F_l)\big) \cdot \chi(F_l).\end{aligned}$$

§ 4. Equivariant Vector Fields and the Transfer Map of Becker–Gottlieb

In this section we shall generalize the localization property for the transfer map to the case of an arbitrary fiber bundles with smooth fiber. The known results [5], [11] are the particular cases of our investigation. In this connection we show the way to obtain these results in terms of the vector field index introduced above.

We consider smooth principal G-bundle $(\mathcal{E}, G, B, p)$ whose base space is a compact connected manifold, and G is a compact Lie group. Let F denote a compact closed connected manifold equipped with a smooth G-action. We assume that there is a tangent Morse–Bott vector field s, that is invariant under G-action, on F . Zeros of the vector field s represent themselves the union of irreducible invariant submanifolds of the manifold F. Let us denote them by $F_1, \dots, F_m$. Because of irreducibility the following manifolds are connected

$$E_l = \mathcal{E} \times_G F_l.$$

Every invariant under G-action tangent Morse–Bott vector field s on F allows us to construct a vector field on $E = \mathcal{E} \times_G F$, that is tangent along the fibers of the bundle E. We denote it also by s. We have

$$s\colon E \to \tau_F(E) = \mathcal{E} \times_G \tau(F).$$

More over the vector field s on E is also a Morse–Bott field. Zeros of the field s on E represent themselves subbundles $E_l = \mathcal{E} \times_G F_l$, $l = 1, \dots, m$, of the bundle E. Let us fix an embedding of the bundle E in $\mathbb{R}^n$. Exactly as it was in the previous section the vector field s defines the maps

$$j_l\colon S^n \wedge E_l^+ \to S^n \wedge E_l^+.$$

Let $\tau(p_l)$ denote the transfer map for the bundle E_l. Let $\hat{g}$ be the map sending $S^n \wedge B^+$ into the union of m spaces $S^n \wedge B^+$ which is induced by the map $g\colon S^n \to \bigvee_m S^n$.

Theorem 4.1. *The transfer map $\tau(p)$ for the bundle E is homotopic to the composition of the maps*

$$\tau(p) \sim i \circ \big(j_1 \circ \tau(p_1) \vee \dots \vee j_l \circ \tau(p_l)\big) \circ \hat{g}$$

where i is an embedding of the Thom space of the trivial n-dimensional vector bundle over $E_1 \cup \dots \cup E_m$ into the Thom space of the trivial n-dimensional vector bundle over E.

Proof. Since the vector field s is invariant respectively to the action of the group G on F, all the homotopies described in the proof of Theorem 3.1 are equivariant accordingly to the action of the structural group on the fiber. Thus, the equivariant analog of Theorem 3.1 holds. As a matter of fact this analog is the formulation of Theorem 4.1.

Remark 4.1. The conditions on the base space of the bundle E can be weakened. We may only require the base to be a cell complex. In this case it is enough to carry on the proof for the fiber of the bundle and then use its equivariantness respectively to the action of the structural group of the bundle. Let us note that homotopy equivalence for an infinite cell complex is understood as equivalence of the map's stable classes.

Let $h^*(\cdot)$ be a multiplicative cohomology theory with unit. We denote by $\big(\mathrm{Ind}_s(E_l)\big)_h$ the image of $\mathrm{Ind}_s(E_l)$ under the canonical transformation of cogomology theories $\mu_h\colon \pi_S^* \to h^*$.

Theorem 4.2. *Under the conditions described above the following formula holds*

$$\tau(p)^* a = \sum_{i=1}^{m} \tau(p_l)^* \big(\big(\mathrm{Ind}_s(E_l)\big)_h \cdot i_l^* a\big)$$

where $a \in h^*(E^+)$ and $i_l\colon E_l \subset E$ is an embedding of the bundles, $l = 1, \dots, m$.

Proof. Let us notice that the maps j_l, as elements from the group of stable-equivalent maps from E_l into itself, uniquely determine the gomomorphism j_l^* of the cogomology groups of the corresponding spaces. Directly from the definition of multiplication in multiplicative cohomology theories it follows that the gomomorphism j_l^* is multiplication on $\big(\mathrm{Ind}_s(E_l)\big)_h$. Thus, the statement of Theorem 4.2 follows from Theorem 4.1.

Theorem 4.2 allows us to obtain the result of [5] in the following form.

Corollary 4.1. *For gomomorphisms in the singular cogomology theory the following equality holds*

$$\tau(p)^* = \sum_{l=1}^{m} \epsilon\big(\mathrm{Ind}_s(E_l)\big)\tau(p_l)^* i^*$$

where $i\colon E_l \subset E,\ \ l = 1, \dots, m$, are embeddings of zero subbundles of the field s.

To obtain the result of [11] from Theorem 4.2 we shall start with discussion of necessary definitions.

Let us consider the principle G-bundle $(\mathcal{E}, G, B, p)$, where G is the compact Lie group. Let G act on a smooth closed compact manifold F.

Definition 4.1. We shall say that *orbits of points $x_1, x_2 \in F$ belong to the same type* if their stabilizers $N(x_1), N(x_2)$ are conjugated. The orbit type of the point x_1 is less then that of the point x_2 if $N(x_2)$ is conjugated to a subgroup in the group $N(x_1)$.

The union of the points belonging to the orbits of the same type γ forms a smooth submanifold F_γ in F whose boundary belongs to the orbit of a smaller type. Let us consider the factor space $Y = F/G$. The closure of the set $Y_\gamma = F_\gamma/G \subset Y$ is a smooth manifold with angles and its boundary lies in $\bigcup_{\gamma' < \gamma} Y_{\gamma'}$ [18].

Theorem 4.3 [18]**.** *There is a simplicial subdivision of space Y, for which the interior of each simplex lies in $Int(Y_\gamma)$ for some γ.*

Let us show how one can prove Theorem 5.14 from [11] using Theorem 4.1.

Theorem 4.4. *Under the conditions described above in an arbitrary cogomology theory for a gomomorphism induced by the transfer map of the bundle $\mathcal{E}\times_G F$, the following formula holds*

$$\tau(p)^* = \sum_{\sigma\subset Y}(-1)^{\dim\sigma}\tau(p_\sigma)^*$$

where $\tau(p_\sigma)$ is the transfer map $\mathcal{E}\times_G\big(G/N(x_\sigma)\big)$, x_σ is an arbitrary preimage of a simplex $\sigma\subset Y$ barycenter in the simplicial subdivision constructed in Theorem 4.3.

Proof. Let us construct the canonical tangent Morse–Bott vector field on the manifold F. To accomplish this at first we shall construct a Morse vector field on each n-dimensional skeleton sk^nY in the simplicial subdivision of the space Y. The construction will be held by induction on skeleton's dimension. Assume that zeroes of the field s are all simplex vertices of the simplicial subdivision. Define the vector field s on sk^1Y to be repelling from sk^0Y and attracting to the barycenter of one-dimensional simplexes. Assume that we have constructed the vector field on sk^nY. Extend it to the field on $\mathrm{sk}^{n+1}Y$ supposing barycenters of $(n+1)$-dimensional simplexes to be attracting zeroes of the field s. This completes both the induction step and the construction of the field. Thus, we have constructed the Morse vector field on each skeleton sk^nY, whose zeroes are all the barycenters of the simplexes. A zero index in the barycenter of a n-dimensional simplex is equal to $(-1)^n$. The map $g\colon F\to F/G=Y$ induces the trivial bundle $g\colon g^{-1}\big(\mathrm{Int}(\sigma)\big)\to\mathrm{Int}(\sigma)$ for any simplex $\sigma\subset Y$. Fix a Riemannian metric, invariant relatively to G, on the manifold F. The canonical splitting of the tangent bundle

$$\tau\big(g^{-1}\big(\mathrm{Int}(\sigma)\big)\big) = g^*\tau\big(\mathrm{Int}(\sigma)\big)\oplus\mathrm{Ker}\,g_*$$

allows one to lift the vector field s on the manifold $g^{-1}\big(\mathrm{Int}(\sigma)\big)$. This lifting will be compatible for different simplexes since

$$\partial g^{-1}(\sigma) = g^{-1}(\partial\sigma).$$

Let us denote by $\hat{s}$ the lifting of the field s on the whole F. The vector field $\hat{s}$ is the Morse–Bott vector field on F that is invariant respectively to the action of the group G on F. Submanifolds $G/N(x_\sigma)$ are zeroes of the vector field $\hat{s}$. We have

$$\mathcal{E}\times_G g^{-1}(\sigma)\cong\big(\mathcal{E}\times_G G/N(x_\sigma)\big)\times\sigma.$$

This means the normal bundle of an embedding

$$\big(\mathcal{E}\times_G G/N(x_\sigma)\big)\subset\mathcal{E}\times_G F$$

is a trivial vector bundle. Thus, $\mathrm{Ind}_{\hat{s}}\big(G/N(x_\sigma)\big)=(-1)^n$, where n is the dimension of the simplex σ. Hence, the statement of the theorem follows.

Example 4.1 (compare [10]). Let us consider a fiber bundle of the Klein bottle KL over the circle S^1 with fiber S^1

$$p\colon KL \xrightarrow{S^1} S^1.$$

The index $I(p)$ of the bundle constructed is equal to the generator of the group

$$Z_2 \cong \pi_1^S \subset \pi_S^0(S^{1+}).$$

Proof. Let ξ_1 be a nontrivial one-dimensional real vector bundle over S^1. Then the bundle $p\colon KL \to S^1$ is isomorphic to the bundle $p\colon RP(\xi_1 \oplus 1) \to S^1$. The structural group of this bundle acts on the fiber S^1 by reflections respectively to the line fixed. The fixed points of the action correspond to two subbundles $RP(\xi_1)$ and $RP(1)$ in the bundle $RP(\xi_1 \oplus 1)$. The normal bundles of embeddings $RP(\xi_1)$, $RP(1) \subset RP(\xi_1 \oplus 1)$ are isomorphic to the bundle $\xi_1 \otimes 1 \cong \xi_1$. Let us construct the vector field on S^1 that is invariant respectively to the action of the structural group of the bundle $RP(\xi_1 \oplus 1)$. The fixed points of the action are zeroes of this vector field. The field is repelling from one zero and attracting to another one. Besides, the vector field is symmetric relatively to the line passing through the fixed points of the action. This field defines the fiber Morse–Bott vector field s on $RP(\xi_1 \oplus 1)$. Zeroes of this vector field are the subbundles $RP(\xi_1)$ and $RP(1)$, whose bundle spaces are the sections of the bundle $RP(\xi_1 \oplus 1)$, that is, S^1. The localization property (Theorem 4.1) applied to the vector field s gives

$$I(p) = \mathrm{Ind}_s\big(RP(\xi_1)\big) + \mathrm{Ind}_s\big(RP(1)\big).$$

The repelling zero of the field s has the index that is equal to 1. The attracting zero has the index as in Example 3.2, that is, $-1+u$, where u is the generator of the group π_1^S. Thus, $I(p) = u$.

§ 5. Morse–Bott Vector Fields on Grassmannizations of Splittable Vector Bundles

In this section we shall construct the fiber Morse–Bott vector field on grassmannization of the splittable vector bundle and shall explicitly calculate its indices. Properties of this vector field play a significant role while proving addition formulae for Pontrjagin characteristic classes.

Below we shall denote by RG_k^n the Grassmannian manifold of k-dimensional subspaces of a n-dimensional real vector space. Consider a n-dimensional real vector bundle η over a base B. We denote by $RG_k^n(\eta)$ the bundle associated to η with fiber RG_k^n.

Let ξ, ζ be real vector bundles of dimensions n_1, n_2, respectively, over B, $(n_1 + n_2 = n)$. Let us define the action of the positive real group $\mathbb{R}_+$ on $\xi \times \zeta$ by the formula $t \circ (v, u) = (v, tu)$. This action is canonically extended on grassmannization $RG_k^n(\xi \times \zeta)$. We define the vector field s by

$$s(z) = \frac{d}{dt}\bigg|_{t=1} t \circ z$$

where $z \in RG_k^n(\xi \times \zeta)$. Let us show that this is a Morse–Bott vector field. It is enough to check that on the typical fiber $RG_k^n(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ of the bundle $RG_k^n(\xi \times \zeta)$ this field is a Morse-Bott field.

Zeroes of restriction of the vector field s on $RG_k^n(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are fixed points of the action $\mathbb{R}_+$ on this fiber, that is, the union of submanifolds $RG_{k_1}^{n_1}(\mathbb{R}^{n_1}) \times RG_{k_2}^{n_2}(\mathbb{R}^{n_2})$, $k_1 + k_2 = k$ (see [7]).

In a neighborhood of a point $z_0 \in RG_{k_1}^{n_1}(\mathbb{R}^{n_1}) \times RG_{k_2}^{n_2}(\mathbb{R}^{n_2}) \subset RG_k^n(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ one can introduce the coordinate system

$$A(z) = \pi_z^{\perp} \circ \pi_z^{-1}$$

where $A(z)$ is a matrix of dimension $(n-k) \times k$; π_z is the orthogonal projection of the plane z onto the plane z_0; $\pi_z^{\perp}$ is the orthogonal projection of the plane z on the orthogonal complement to z_0 (the neighborhood of the point z_0 consists of all points z for which π_z is isomorphizm). Since the plane z_0 is the direct sum of two planes $x_0 \in RG_{k_1}^{n_1}(\mathbb{R}^{n_1})$ and $y_0 \in RG_{k_2}^{n_2}(\mathbb{R}^{n_2})$, the matrix $A(z)$ is canonically divided into the blocks

$$A(z) = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

where D_1 is a matrix of dimension $(n_1 - k_1) \times k_1$, D_2 has dimension $(n_1 - k_1) \times k_2$, D_3 has dimension $(n_2 - k_2) \times k_1$, D_4 has dimension $(n_2 - k_2) \times k_2$. The points of the submanifold $RG_{k_1}^{n_1}(\mathbb{R}^{n_1}) \times RG_{k_2}^{n_2}(\mathbb{R}^{n_2})$ in this neighborhood are extracted by the equations $D_2 = D_3 = 0$, and the tubular neighborhood fiber's points of the same submanifold over the point z_0 are extracted by the equations $D_1 = D_4 = 0$. In this neighborhood of the point z_0 the group $\mathbb{R}_+$ acts by the following way

$$t \circ \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} D_1 & D_2/t \\ tD_3 & D_4 \end{pmatrix}. \tag{5.1}$$

Hence, the vector field constructed is a Morse–Bott vector field.

Let us describe behavior of the vector field s on the whole space $RG_k^n(\xi \times \zeta)$. Let $\mathcal{E}$ be the principal $O(n)$-bundle associated with the vector bundle $\xi \times \zeta$ over B. Zeroes of the field s are subbundles $RG_{k_1}^{n_1}(\xi) \times RG_{k_2}^{n_2}(\zeta)$, $k_1 + k_2 = k$. The bundle of tangents along the fibers of the bundle $RG_k^n(\xi \times \zeta)$ is the vector bundle

$$\mathcal{E} \times_{O(n)} \tau(RG_k^n).$$

Let $\nu(RG_{k_1}^{n_1} \times RG_{k_2}^{n_2})$ be the normal bundle of the embedding $RG_{k_1}^{n_1} \times RG_{k_2}^{n_2} \subset RG_k^n$. It is isomorphic to the direct sum of the vector bundles $\xi_{n_1-k_1} \otimes \xi_{k_2}$ and $\xi_{k_1} \otimes \xi_{n_2-k_2}$, where we denote by ξ_k the k-dimensional tautological vector bundle over RG_k^n; ξ_{n-k} is the $(n-k)$-dimensional vector bundle over $RG_{k_2}^{n_2}$ that is complementary to ξ (remark that the bundle ξ_{n-k} is isomorphic to the $(n-k)$-dimensional tautological vector bundle over RG_{n-k}^n). Thus, the normal bundle of the embedding

$$RG_{k_1}^{n_1}(\xi) \times RG_{k_2}^{n_2}(\zeta) \subset RG_k^n(\xi \times \zeta)$$

is the bundle

$$\nu = \mathcal{E} \times_{O(n)} \nu(RG^{n_1}_{k_1} \times RG^{n_2}_{k_2}),$$

and

$$\nu \cong \pi_1^*\xi(k_1) \otimes \pi_2^*\zeta(n_2 - k_2) \oplus \pi_1^*\xi(n_1 - k_1) \otimes \pi_2^*\zeta(k_2)$$

where we denote by $\xi(k)$ the k-dimensional tautological vector bundle over $RG^n_k(\xi)$ (respectively, $\xi(n-k)$ is the $(n-k)$-dimensional tautological vector bundle over $RG^n_{n-k}(\xi) \cong RG^n_k(\xi)$); π_1, π_2 are the projections on the first and the second factors in the product $RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)$.

Lemma 5.1. *The vector field s associates the map (see § 3)*

$$\psi^s_{k_1 k_2} : \nu \to \nu$$

to the subbundle $RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)$ by the formula

$$\phi^s_{k_1 k_2}(u, v) = (u, -v)$$

where $u \in \pi_1^*\xi(k_1) \otimes \pi_2^*\zeta(n_2 - k_2)$, $v \in \pi_1^*\xi(n_1 - k_1) \otimes \pi_2^*\zeta(k_2)$.

Proof. It is enough to verify this statement on each fiber of the bundle $RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)$. Owing to (5.1) on each fiber of this bundle restriction of $\phi^s_{k_1 k_2}$ on the fiber is the symmetry respectively to the subbundle $\xi_{k_1} \otimes \xi_{n_2 - k_2}$. Hence, the lemma's statement follows.

In the previous notation we have the following statments.

Corollary 5.1. *In the case $k_2 = 0$ the index of the vector field s on the subbundle of zeroes $RG^n_k(\xi) \times B$ is equal to 1.*

The proof is obvious.

Corollary 5.2. *The square of the index $\mathrm{Ind}_s\big(RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)\big)$ of the vector field s zero is equal to 1 for any k_1 and k_2 whenever $k_1 + k_2 = k$.*

Proof. Let us remark that $(\psi^s_{k_1 k_2})^2 = \mathrm{id}$ for all k_1 and k_2 such that $k_1 + k_2 = k$. Thus, the statement of the corollary follows from the definition of the multiplication in the ring $\pi^0_S(\cdot)$ as a composition of the maps.

Corollary 5.3. *The following equality holds*

$$\epsilon\big(\mathrm{Ind}_s\big(RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)\big)\big) = (-1)^{(n_1 - k_1)k_2}.$$

The proof is obvious.

In the end of this section we shall calculate the index of the zero of the vector field s in the case when the corresponding map ψ^s is symmetry respectively to a certain subbundle. As a corollary we shall get an explicit expression for zero indices of the vector field s constructed in this section.

In [6] the cobordism theory $CO^*(\cdot)$ constructed using manifolds, whose stable tangent bundle is endowed with a fixed structure of complexification of the real vector bundle, was considered. Any bundle of the type $\mathbb{C}\otimes\xi$ is oriented in this theory, that is, it possesses the Thom class. For any n-dimensional vector bundle ξ the canonical Thom class of the bundle $\mathbb{C}\otimes\xi$ in this theory is defined using the classified map

$$u(\xi)\colon T(\mathbb{C}\otimes\xi)\to T(\mathbb{C}\otimes\xi_n)$$

where ξ_n is the n-dimensional tautological vector bundle over $BO(n)$. Let us denote by $\bar{u}(\xi)$ another Thom class of the bundle $\mathbb{C}\otimes\xi$ defined in this theory by the map

$$\bar{u}(\mathbb{C}\otimes\xi): T(\mathbb{C}\otimes\xi)=T(\xi\oplus\xi)\overset{(1,-1)}{\longrightarrow}T(\xi\oplus\xi)=T(\mathbb{C}\otimes\xi)\overset{u(\xi)}{\longrightarrow}T(\mathbb{C}\otimes\xi_n).$$

Let ξ be a real vector bundle over a finite cell complex B and $\xi^\perp$ be its orthogonal complement. Let us construct the map

$$\rho(\xi)\colon T(\xi\oplus\xi^\perp)\overset{(-1,1)}{\longrightarrow}T(\xi\oplus\xi^\perp).$$

We denote by σ_m the suspension isomorphism $CO^*(B^+)\to CO^{*+m}(S^m\wedge B^+)$.

Theorem 5.1. *For the real vector bundle ξ in the ring $CO^*(B^+)$ the following formula holds*

$$\sigma_k^{-1}\rho(\xi)^*\sigma_k(1)=u(\xi)^{-1}\bar{u}(\xi) \tag{5.2}$$

where we denote by u^{-1} the inverse map to the Thom isomorphism in the theory $CO^*(\cdot)$ induced by the Thom class u, where $k=\dim(\xi\oplus\xi^\perp)$.

Proof. Remark that the map $\Sigma^k\big(\rho(\xi)\big)$is homotopic to the map

$$T(\xi\oplus\xi\oplus\xi^\perp\oplus\xi^\perp)\xrightarrow{(1,-1,1,1)}T(\xi\oplus\xi\oplus\xi^\perp\oplus\xi^\perp).$$

Thus, the statement of the theorem immediately follows from multiplicativity of the Thom class.

We shall denote below by $\gamma(\xi)$ the element (5.2) of the ring $CO^*(B^+)$.

Suppouse F_l is a connected component of the zero submanifold of the Morse–Bott field s on the manifold F, and the corresponding map ψ^s is a symmetry respectively to a certain subbundle. Now let ξ be the subbundle on which ψ^s acts by fiber multiplication by -1.

Corollary 5.4. *The following formula holds*

$$\big(\mathrm{Ind}_s(F_l)\big)_{CO}=\gamma(\xi).$$

In particular, for the vector field s on grassmannization of the splittable vector bundle constructed above in this section the equality holds

$$\big(\mathrm{Ind}\big(RG^{n_1}_{k_1}(\xi)\times RG^{n_2}_{k_2}(\zeta)\big)\big)_{CO}=\gamma\big(\pi_1^*\big(\xi(n_1-k_1)\big)\otimes\pi_2^*\big(\zeta(k_2)\big)\big).$$

Let us consider the formal group $f(u,v)=u+v+\sum_{i,j\geqslant 1}\alpha_{ij}u^iv^j$, where $\alpha_{ij}\in\Omega_U$ in complex cobordisms [15]. Let $\bar u\in U^2\bigl(CP(\infty)\bigr)$ be the inverse element for $u\in U^2\bigl(CP(\infty)\bigr)$ in this group, that is, such a formal row in $\Omega_U\bigl[[u]\bigr]$ that $f(u,\bar u)=0$. Denote by $\phi(x)$ the row such that $\bar u=u\phi(u)$, and $\phi_n(c_1,\dots,c_n)$ the row in the ring $U^*\bigl(BU(n)\bigr)\cong\Omega_U\bigl[[c_1,\dots,c_1]\bigr]$ defined by the identity

$$\phi_n\bigl(\sigma_1(x_1,\dots,x_n),\dots,\sigma_n(x_1,\dots,x_n)\bigr)=\prod_{i=1}^{n}\phi(x_i)$$

where σ_i are elementary symmetric polynomials of n variables.

Using the canonical transformation $\mu_U^{CO}\colon CO^*(\cdot)\to U^*(\cdot)$, immediately from Theorem 5.1 we get.

Corollary 5.5. *In the ring of complex cobordisms $U^*(B^+)$ the following formula holds*

$$\bigl(\mathrm{Ind}_s(F_l)\bigr)_U=\gamma_U(\xi)=\phi_n\bigl(c_1(\mathbb{C}\xi),\dots,c_n(\mathbb{C}\xi)\bigr)$$

where $\gamma_U(\xi)=\mu_U^{CO}\bigl(\gamma(\xi)\bigr)$.

Corollary 5.6. *For a one-dimensional real vector bundle ξ in the complex cobordism ring $U^*(B^+)$ the following formula holds*

$$\gamma_U(\xi)=-1-\sum_{i,j\geqslant 1}\alpha_{ij}\bigl(c_1(\mathbb{C}\xi)\bigr)^{i+j-1}.$$

Proof. Remark that $c_1(\mathbb{C}\xi)=c_1(\,\overline{\mathbb{C}\xi}\,)$. Thus,

$$f\bigl(c_1(\mathbb{C}\xi),c_1(\mathbb{C}\xi)\bigr)=0.$$

Hence, the desired result is obtained.

§ 6. Addition Theorems for Pontrjagin Characteristic Classes

In this section we shall apply the localization property of the transfer map for studying characteristic classes of the real vector bundles introduced in [6]. In case of ordinary integral cogomologies integer classes coincide with classical Pontrjagin characteristic classes and that is why they were called in [6] as Pontrjagin classes. The construction of characteristic classes [6] is based on the following properties of the transfer map (see [3], [8]).

Let (E_i,F_i,B_i,p), $i=1,2$, be fiber bundles whose fibers are smooth compact closed manifolds.

Property 6.1 (functoriality). *If*

$$\psi\colon (E_1,F_1,B_1,p)\to(E_2,F_2,B_2,p)$$

is a bundle map, then $\tau(p_2)\circ\psi\sim\psi\circ\tau(p_1)$.

Property 6.2 (multiplicativity). *If*

$$(E, F, B, p) = (E_1 \times E_2, F_1 \times F_2, B_1 \times B_2, p_1 \times p_2),$$

then $\tau(p_1) \wedge \tau(p_2) \sim \tau(p_1 \times p_2)$.

Let ξ be a real n-dimensional vector bundle over a finite cell complex B (we can assume without loss of generality that B is a smooth manifold). Let us consider grassmannization of the vector bundle ξ , that is, the bundle

$$\big(RG_k^n(\xi), RG_k^n, B, p_k\big).$$

for given k, $1 \leq k \leq n$. Let $\tau(p_k)$ be the transfer map for this bundle. We denote by $\xi(k)$ the tautological vector bundle over $RG_k^n(\xi)$ and by f_k its classified map

$$\xi(k) = f_k^* \xi_k$$

where ξ_k is the tautological bundle over $BO(k)$. Let

$$s_0 \colon BO(k) \to T(\mathbb{C} \otimes \xi_k)$$

be the embedding on the zero section.

Definition 6.1. In the theory CO^* the half-integer Pontrjagin class $p_{k/2}(\xi)$ is equal to

$$p_{k/2}(\xi) = \tau(p_k)^* f^* \chi(\mathbb{C} \otimes \xi_k) = \tau(p_k)^* f^* [s_0] \tag{6.1}$$

where $\chi(\mathbb{C} \otimes \zeta)$ is the Euler class of the bundle $\mathbb{C} \otimes \zeta$ in the theory CO^* (see [6]). Suppose $p_{k/2}(\xi) = 0$ for $k > n$.

As it follows from the property of the transfer map 6.1 the formula (6.1) defines characteristic classes of real vector bundles.

Theorem 6.1. *Let ξ and ζ be real vector bundles* ($\dim \xi = n_1$, $\dim \zeta = n_2$, $n_1 + n_2 = n$). *Then the following formula holds*

$$p_{k/2}(\xi \times \zeta) = \sum_{k_1 + k_2 = k} \tau(p_{k_1 k_2})^* \Big(\gamma\big(\pi_1^* \xi(n_1 - k_1) \otimes \pi_2^* \zeta(k_2)\big) \chi\big(\mathbb{C} \otimes \xi(k_1)\big) \\ \times \chi\big(\mathbb{C} \otimes \zeta(k_2)\big)\Big)$$

for any k, where $\tau(p_{k_1 k_2})$ is the transfer map of the bundle $RG_{k_1}^{n_1}(\xi) \times RG_{k_2}^{n_2}(\zeta)$, π_1 and π_2 are the projections on the first and second factors in the product $RG_{k_1}^{n_1}(\xi) \times RG_{k_2}^{n_2}(\zeta)$.

Proof. Let us apply the localization property of the transfer map in the form of Theorem 4.2 to the vector field s on $RG_k^n(\xi \times \zeta)$ constructed in § 5. Owing to Corollary 5.4 we obtain

$$\tau(p_k)^* \chi\big(\mathbb{C} \otimes \eta(k)\big) = \sum_{k_1 + k_2 = k} \tau(p_{k_1 k_2})^* \Big(\gamma\big(\pi_1^* \xi(n_1 - k_1) \otimes \pi_2^* \zeta(k_2)\big) \chi\big(\mathbb{C} \otimes \xi(k_1)\big) \\ \times \chi\big(\mathbb{C} \otimes \zeta(k_2)\big)\Big) \tag{6.2}$$

where we denote by $\eta(k)$ the k-dimensional tautological vector bundle over $RG_k^n(\xi \times \zeta)$. It remains to remark that the left part of (6.2) is equal to $p_{k/2}(\xi \times \zeta)$ by the definition.

Let us show the method to obtain from Theorem 6.1 the addition formula for Pontrjagin classes modulo elements of 2-primary order.

Theorem 6.2. *Let ξ and ζ be real vector bundles over a finite cell base B $(\dim\xi = n_1,\ \dim\zeta = n_2,\ n_1 + n_2 = n)$. For any k the following formula holds*

$$p_{k/2}(\xi \oplus \zeta) = \sum_{k_1+k_2=k} (-1)^{(n_1-k_1)k_2} p_{k_1/2}(\xi) p_{k_2/2}(\zeta) \pmod{2\,\mathrm{Tors}}.$$

Proof. Owing to functoriality of the Pontrjagin classes it is enough to prove the statement of the theorem for the bundle $\pi_1^*(\xi) \oplus \pi_2^*(\zeta)$ over $B \times B$, where π_1 and π_2 are the projections on the first and the second factors in the product $B \times B$. Remark also that

$$\pi_1^*(\xi) \oplus \pi_2^*(\zeta) \cong \xi \times \zeta.$$

Assume that

$$\alpha = \big(\mathrm{Ind}_s\big(RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)\big)\big)_{CO}.$$

¿From Corollary 5.2 $\alpha^2 = 1$. We have $\alpha = \pm 1 + u$, where $u \in \widetilde{CO}^0\big(RG^{n_1}_{k_1}(\xi) \times RG^{n_2}_{k_2}(\zeta)^+\big)$. Since we work in the category of finite cell complexes, the element u is nilpotent. The relation $\alpha^2 = 1$ involves the equality $\pm 2u + u^2 = 0$. Hence and from nilpotency of u we conclude that the element u is 2-primary, that is, $u \in 2\,\mathrm{Tors}$. Thus, $\alpha = \pm 1(\mathrm{mod}(2\,\mathrm{Tors}))$. From Corollary 5.3

$$\alpha \equiv (-1)^{(n_1-k_1)k_2} \pmod{2\,\mathrm{Tors}}. \tag{6.3}$$

We denote by $\eta(k)$ the k-dimensional tautological vector bundle over $RG^n_k(\xi \times \zeta)$. From the formula (6.2) taking into account (6.3) we obtain

$$\begin{aligned}\tau(p_k)^*\chi\big(\mathbb{C} \otimes \eta(k)\big) &= \\ &= \sum_{k_1+k_2=k} (-1)^{(n_1-k_1)k_2} \tau(p_{k_1k_2})^*\big(\chi(\mathbb{C} \otimes \xi_{k_1}) \times \chi(\mathbb{C} \otimes \zeta_{k_2})\big) \pmod{2\,\mathrm{Tors}}.\end{aligned} \tag{6.4}$$

¿From multiplicativity of the transfer map 6.2 it follows that

$$\begin{aligned}&\tau(p_{k_1} \times p_{k_2})^*\big(\chi\big(\mathbb{C} \otimes \xi(k_1)\big) \times \chi\big(\mathbb{C} \otimes \zeta(k_2)\big)\big) \\ &\quad = \tau(p_{k_1})^*\chi\big(\mathbb{C} \otimes \xi(k_1)\big)\tau(p_{k_2})^*\chi\big(\mathbb{C} \otimes \zeta(k_2)\big).\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5) we obtain the formula required

$$p_{k/2}(\xi \times \zeta) = \sum_{k_1+k_2=k} (-1)^{(n_1-k_1)k_2} p_{k_1/2}(\xi) \times p_{k_2/2}(\zeta) \pmod{2\,\mathrm{Tors}}.$$

Theorem 6.3. *Half-integer Pontrjagin classes are stable*

$$p_{k/2}(\xi \oplus 1) = p_{k/2}(\xi) = p_{k/2}(1 \oplus \xi).$$

Proof. Let us apply the localization property of the transfer map to the bundle $RG_k^{n+1}(\xi \oplus 1)$ $(\dim \xi = n)$. Using Corollary 5.1 we obtain that for any element $x \in CO^*\big(RG_k^{n+1}(\xi \oplus 1)^+\big)$ and for a certain element $\alpha \in CO^0\big(RG_k^{n+1}(\xi \oplus 1)^+\big)$ the following splitting holds

$$\tau(p)^* x = \tau(p_1)^* i_1^* x + \tau(p_2)^* \alpha i_2^* x$$

where $\tau(p_1)$ is the transfer map of the bundle $\big(RG_k^n(\xi), RG_k^n, B, p_1\big)$ and $\tau(p_2)$ is the transfer map of the bundle $\big(RG_{k-1}^n(\xi), RG_{k-1}^n, B, p_2\big)$, the other maps were defined in the formulation of the localization property.

Since under the embedding $RG_{k-1}^n(\xi) \subset RG_k^{n+1}(\xi \oplus 1)$ the tautological vector bundle $\xi(k)$ over $RG_k^{n+1}(\xi \oplus 1)$ turns to the sum $\xi(k-1) \oplus 1$, where $\xi(k-1)$ is the tautological vector bundle over $RG_{k-1}^n(\xi)$, then

$$\begin{aligned}\tau(p)^* \chi\big(\otimes \xi(k)\big) &= \tau(p_1)^* \chi\big(\otimes \xi(k)\big) + \tau(p_2)^* \alpha \chi\big(\otimes \big(\xi(k-1) \oplus 1\big)\big) \\ &= \tau(p_1)^* \chi\big(\otimes \xi(k)\big).\end{aligned}$$

Hence, we obtain the first equality of the theorem.

To prove the second equality it is enough to multiply the vector field from § 5 by -1 and to proceed with an analogous reasoning.

Corollary 6.1. *The half-integer Pontrjagin class $p_{k/2}(\xi)$ is 2-primary for odd k.*

Proof. Using Theorem 6.2

$$p_{k/2}(1 \oplus \xi) = (-1)^k p_{k/2}(\xi) \pmod{2\,\mathrm{Tors}},$$

and from Theorem 6.3

$$p_{k/2}(1 \oplus \xi) = p_{k/2}(\xi).$$

For odd k these equalities are compatible only if $p_{k/2}(\xi) \in 2\,\mathrm{Tors}$.

Corollary 6.2. *For any two real vector bundles ξ and ζ the following relation holds* [6]

$$p_{2k/2}(\xi \oplus \zeta) = \sum_{k_1+k_2=k} p_{2k_1/2}(\xi) p_{2k_2/2}(\zeta) \pmod{2\,\mathrm{Tors}}.$$

Theorem 6.4. *For the image of Pontrjagin characteristic classes in the complex cobordisms there is a formula, which expresses the half-integer Pontrjagin classes of the sum of two real vector bundles in terms of characteristic classes of their summands.*

Proof. Owing to splitting (6.2) it is enough to prove that $\big(\mathrm{Ind}_s\big(RG^{n_1}_{k_1}(\xi)\times RG^{n_2}_{k_2}(\zeta)\big)\big)_U$ can be expressed as a polynomial of characteristic classes of the bundles $\xi(k_1)$ and $\zeta(k_2)$ with coefficients in $p_1^*\big(U^*(B)\big)\otimes p_2^*\big(U^*(B)\big)$, where p_1 and p_2 are the projections in the bundles $RG^{n_1}_{k_1}(\xi)$ and $RG^{n_2}_{k_2}(\zeta)$ respectively. Owing to Lemma 5.1 and Corollary 5.5 the following equality holds in complex cobordisms

$$\big(\mathrm{Ind}_s\big(RG^{n_1}_{k_1}(\xi)\times RG^{n_2}_{k_2}(\zeta)\big)\big)_U=\phi_{(n_1-k_1)k_2}\big(c_1(\eta),\dots,c_{(n_1-k_1)k_2}(\eta)\big)$$

where $\eta\cong\mathbb{C}\otimes\big(\xi(n_1-k_1)\otimes\zeta(k_2)\big)$. Using the splitting principle for complex vector bundles and the formula for the formal group

$$f\big(c_1(\eta_1),c_1(\eta_2)\big)=c_1(\eta_1\otimes\eta_2)$$

we obtain the expression of the classes $c_i(\eta)$ in terms of characteristic classes of the bundles $\mathbb{C}\otimes\xi(n_1-k_1)$ and $\mathbb{C}\otimes\zeta(k_2)$. Finally, it is enough to remark that $\xi(n_1-k_1)\oplus\xi(k_1)\cong p_1^*\xi$.

Remark 6.1. Half-integer Pontrjagin classes $p_{k/2}(\xi)$ with odd numbers k can have any 2-primary order. For example, let ξ_1 be the one-dimensional tautological vector bundle over RP^{2n+1}. We denote by $\mu^{CO}_K\colon CO^*(\cdot)\to K^*(\cdot)$ the canonical transformation of the cogomology theories. The order of $\mu^{CO}_K\big(p_{1/2}(\xi_1)\big)\in K^*(RP^{2n+1})$ is equal to 2^n.

At the end of this section we shall consider in detail the addition theorem for the first half-integer Pontrjagin class (see [9]).

We assume that ξ and ζ are vector bundles over a finite cell base B. Let $\tau(p)$ be the transfer map of the bundle $RP(\xi\oplus\zeta)$, $\tau(p_1)$ the transfer map of the bundle $RP(\xi)$, $\tau(p_2)$ the transfer map of the bundle $RP(\zeta)$. Let us denote by i_1 the embedding $RP(\xi)\subset RP(\xi\oplus\zeta)$, by i_2 the embedding $RP(\zeta)\subset RP(\xi\oplus\zeta)$.

Theorem 6.5. *The following formula holds*

$$\tau(p)^*(x)=\tau(p_1)^*i_1^*(x)+\tau(p_2)^*\big(\gamma_U\big(p_2^*\xi\otimes\zeta(1)\big)i_2^*(x)\big)$$

for any $x\in U^*\big(RP(\xi\oplus\zeta)\big)$.

Proof. The statement of the theorem immediately follows from Theorem 4.2 and Corollaries 5.1 and 5.4.

Corollary 6.3. *For any two one-dimensional real vector bundles ξ and ζ over a finite cell base the following equality holds*

$$p_{1/2}(\xi\oplus\zeta)=u-v-\sum_{i,j\geqslant 1}\alpha_{ij}f(u,v)^{i+j-1}v$$

where $u = p_{1/2}(\xi)$, $v = p_{1/2}(\zeta)$.

Proof. For any one-dimensional vector bundle ξ we have $p_{1/2}(\xi) = c_1(\mathbb{C}\otimes\xi)$. Thus, the statement required follows from the explicit form $\gamma_U(\xi\otimes\zeta)$ (see Corollary 5.6).

Let $[u]_2 = ua(u)$, where $a(u) = 2 + \sum_{i,j\geqslant 1}\alpha_{ij}u^{i+j-1}$. Then the result of Corollary 6.3 can be written in the form

$$p_{1/2}(\xi\oplus\zeta) = u + v - a\big(f(u,v)\big)v.$$

Next let us introduce the following series, symmetric on u and v:

$$\delta(u,v) = \frac{a(u)-a(v)}{u-v}, \qquad d(u,v) = \frac{va(u)-ua(v)}{u-v}.$$

Let us write the series $\alpha(u) = 1 + \sum_{i\geqslant 1}\alpha_{i1}u^i$ in the form

$$\alpha(u) = \alpha_0(u^2) + u\alpha_1(u^2).$$

Theorem 6.6. *For any two real one-dimensional vector bundles ξ and ζ the following equality holds*

$$p_{1/2}(\xi\oplus\zeta) = u + v - uv\big[\alpha_0(uv)\delta(u,v) + \alpha_1(uv)d(u,v)\big]$$

where $u = p_{1/2}(\xi)$ *and* $v = p_{1/2}(\zeta)$.

Proof. Let

$$\Phi(u,v) = 1 + \sum_{i,j\geqslant 1}\alpha_{ij}u^i\left(\frac{v^j-\bar{u}^j}{v-\bar{u}}\right) \in \Omega_U\big[[u,v]\big]$$

where $\bar{u}$ satisfies the relation $f(u,\bar{u}) = 0$. Then

$$f(u,v) = f(u,v) - f(u,\bar{u}) = (v-\bar{u})\Phi(u,v).$$

We get

$$f\big([u]_2,[v]_2\big) = \big[f(u,v)\big]_2 = f(u,v)a\big(f(u,v)\big).$$

Therefore,

$$\big([v]_2 - [\bar{u}]_2\big)\Phi\big([u]_2,[v]_2\big) = (v-\bar{u})\Phi(u,v)a\big(f(u,v)\big).$$

Thus, in the ring $\Omega_U\big[[u,v]\big]$ the following identity holds

$$\Phi(u,v)a\big(f(u,v)\big)v = \left([v]_2 + \bar{u}v\frac{a(v)-a(\bar{u})}{v-\bar{u}}\right)\Phi\big([u]_2,[v]_2\big).$$

To prove Theorem 6.6 it is enough to assume that ξ and ζ are one-dimensional tautological vector bundles over $RP(\infty)$. Thus, all further computations can be held in the factor ring

$$A = \Omega_U\big[[u,v]\big]/\big([u]_2,[v]_2\big).$$

¿From the identity $[u]_2 = (u-\bar u)\Phi(u,u)$ in $\Omega_U\big[[u]\big]$ immediately follows that in the factor ring $u=\bar u$ and $v=\bar v$. Using the relations $[u]_2=0$ and $u=\bar u$ we obtain the following identity in A

$$\Phi(u,v)a\big(f(u,v)\big)v = uv\,\delta(u,v),$$

from which it immediately follows that in A the following relation holds

$$a\big(f(u,v)\big)v = a\big(f(u,v)\big)u.$$

Using now an explicit form of the series $\Phi(u,v)$ introduced above we obtain that in the ring A the following identity holds

$$a\big(f(u,v)\big)\Phi(u,v) = a\big(f(u,v)\big)\frac{\partial}{\partial v}f(u,v).$$

Lemma 6.1. *In the ring $\Omega_U\big[[u,v]\big]$ the following formula holds*

$$\frac{\partial}{\partial v}f(u,v) = \frac{CP(v)}{CP\big(f(u,v)\big)}$$

where $CP(v) = dg(v)/dv = 1+\sum_{n\geqslant 1}\big[CP(n)\big]v^n$ and $g(v)$ is logarithm of the group $f(u,v)$.

Proof. In the ring $\Omega_U\otimes Q\big[[u,v]\big]$ for the series $f(u,v)$ the following expression holds

$$f(u,v) = g^{-1}\big(g(u)+g(v)\big).$$

Since

$$\frac{dg^{-1}(x)}{dx}\cdot\frac{dg\big(g^{-1}(x)\big)}{dg^{-1}(x)} = 1,$$

we obtain that

$$\frac{\partial}{\partial v}f(u,v) = \frac{\partial g^{-1}\big(g(u)+g(v)\big)}{\partial g(v)}\frac{\partial g(v)}{\partial v} = \frac{CP(v)}{CP\big(f(u,v)\big)}.$$

Returning to the ring A we obtain

$$a\big(f(u,v)\big)\frac{\partial}{\partial v}f(u,v) = a\big(f(u,v)\big)\frac{CP(v)}{CP\big(f(u,v)\big)} = a\big(f(u,v)\big)CP(v).$$

Let us remind that

$$\frac{1}{CP(v)} = \frac{\partial f(u,v)}{\partial u}\bigg|_{u=0} = 1+\sum_{i\geqslant 1}\alpha_{i1}v^i = \alpha(v).$$

Therefore,

$$uv\,\delta(u,v)\alpha(v) = \alpha(v)\Phi(u,v)a\big(f(u,v)\big)v = a\big(f(u,v)\big)v.$$

Lemma 6.2. *In the ring A the following relations hold*

$$uv\,\delta(u,v)v = uv\,d(u,v),$$
$$uv\,\delta(u,v)v^2 = uv\,\delta(u,v)uv.$$

Proof. The first equality follows from the relation

$$uv\frac{a(v)-a(u)}{v-u}v = uv\frac{va(v)-va(u)+ua(v)-ua(v)}{v-u}$$
$$= uv\frac{ua(v)-va(u)+(v-u)a(v)}{v-u} = uv\frac{ua(v)-va(u)}{v-u} = uv\,d(u,v).$$

The second one follows from the relation $a\big(f(u,v)\big)v = a\big(f(u,v)\big)u$:

$$uv\,\delta(u,v)v^2 = \Phi(u,v)a\big(f(u,v)\big)v^3 = \Phi(u,v)a\big(f(u,v)\big)vuv = uv\,\delta(u,v).$$

Applying the relations from Lemma 6.2 we obtain

$$uv\,\delta(u,v)\alpha(v) = uv\,\delta(u,v)\big(\alpha_0(v^2)+v\alpha_1(v^2)\big)$$
$$= uv\big(\alpha_0(uv)\delta(u,v)+\alpha_1(uv)\delta(u,v)v\big) = uv\big(\alpha_0(uv)\delta(u,v)+\alpha_1(uv)d(u,v)\big),$$

that allows us to finish the proof of Theorem 6.6.

This theorem gives an explicit form of the formal series, symmetric on u and v

$$b(u,v) = u+v+\sum_{i,j\geqslant 1}\beta_{ij}u^iv^j$$

such that for any two one-dimensional real vector bundles ξ and ζ over X the following formula holds

$$p_{1/2}(\xi\oplus\zeta) = b(u,v)\in U^2(X)$$

where $u = p_{1/2}(\xi)$ and $v = p_{1/2}(\zeta)$.

Theorem 6.7 (the addition formula for the first half-integer Pontrjagin class). *For any two real vector bundles ξ and ζ*

$$p_{1/2}(\xi\oplus\zeta) = u+v+\sum_{k,l\geqslant 1}\beta_{kl}s_{k-1}(u)s_{l-1}(v)$$

where $u = p_{1/2}(\xi)$, $v = p_{1/2}(\zeta)$, s_k are the Landweber–Novikov operations in complex cobordisms [15].

Proof. ¿From the localization theorem for the transfer map it follows that

$$p_{1/2}(\xi\oplus\zeta) = p_{1/2}(\xi)+\tau(p)^*\big(\gamma_U\big(p^*(\xi)\otimes\zeta(1)\big)\cdot w\big)$$

where $\tau(p)$ is the transfer map of the bundle $RP(\zeta)$, $w=\chi\big(\mathbb{C}\otimes\zeta(1)\big)$. First let us assume that $\dim\xi=1$. We have

$$p_{1/2}(\xi\oplus\zeta)=u+\tau(p)^*\big(\gamma_U\big(p^*\xi\otimes\zeta(1)\big)w\big), \tag{6.6}$$

$$p_{1/2}(\zeta\oplus\xi)=v+\gamma_U(\zeta\otimes\xi)u \tag{6.7}$$

where $u=p_{1/2}(\xi)$, $v=p_{1/2}(\zeta)$.

Remark that from Corollary 5.6

$$\begin{aligned}\tau(p)^*\big(\gamma_U\big(p^*\xi\otimes\zeta(1)\big)w\big)&=-\tau(p)^*\Big(w+\sum_{i,j\geqslant1}\alpha_{ij}f(p^*u,w)^{i+j-1}w\Big)\\&=\tau(p)^*\big(p_{1/2}\big(p^*\xi\oplus\zeta(1)\big)-p_{1/2}(p^*\xi)\big).\end{aligned}\tag{6.8}$$

The Landweber-Novikov operations are stable cohomology operations, therefore, they commute with the map $\tau(p)^*$. Owing to Theorem 6.6 we obtain that the expression (6.8) is equal to

$$\tau(p)^*\Big(w+\sum_{k,l\geqslant1}\beta_{kl}(p^*u)^kw^l\Big)=v+\sum_{k,l\geqslant1}\beta_{kl}u^ks_{l-1}(v).$$

Setting (6.6) to be equal to (6.7) we obtain

$$\gamma_U(\zeta\otimes\xi^*)\cdot u=u+\sum_{k,l\geqslant1}\beta_{kl}u^ls_{k-1}(v).$$

Consider now the case of an arbitrary ξ. We have

$$\begin{aligned}p_{1/2}(\xi\oplus\zeta)&=u+\tau(p)^*\big(\gamma_U\big(p^*(\xi)\otimes\zeta(1)^*\big)w\big)\\&=u+\tau(p)^*\Big(w+\sum_{k,l\geqslant1}\beta_{kl}w^lp^*s_{k-1}(u)\Big)=u+v+\sum_{k,l\geqslant1}\beta_{kl}s_{l-1}(v)s_{k-1}(u).\end{aligned}$$

This completes the proof.

Theorem 6.8. *The characteristic class $p_{1/2}$ defines the map*

$$p_{1/2}\colon KO(X)\to U^2(X),$$

on whose image there is an associative addition operation

$$x\oplus y=x+y+\sum_{k,l\geqslant1}\beta_{kl}s_{l-1}(y)s_{k-1}(x).$$

Let us pay attention that this operation is not reduced to any formal group.

§ 7. Pontrjagin Characteristic Classes of Complex Vector Bundles

Theorem 7.1. *Let η be a n-dimensional complex vector bundle. The transfer map $\tau(p)$ of the bundle $RG^{2n}_{2k+1}(\eta)$ is homotopic to the constant map. The transfer maps $\tau(p_1)$ and $\tau(p_2)$ of the bundles $RG^{2n}_{2k}(\eta)$ and $CG^n_k(\eta)$ are connected by the relation*

$$\tau(p_1) \sim i \circ \tau(p_2)$$

where $i\colon CG^n_k(\eta) \to RG^{2n}_{2k}(\eta)$ is the canonical embedding.

Proof. Let us consider the bundle $RG^{2n}_k(\eta) \xrightarrow{RG^{2n}_k} B$. The group $U(1) = S^1$ acts on the space of this bundle. This action is induced by the complex structure in the vector bundle η. All hyperplanes α such that for any vector $v \in \alpha$ the vector iv belongs to α are fixed points of the action. Since the vectors v and iv are linearly independent over the real number field, the fixed points can appear whenever k is even. Applying the localization property to the vector field corresponding to the action we obtain that the transfer map $\tau(p)$ of the bundle $RG^{2n}_{2k+1}(\eta)$ is homotopic to the constant map (see Remark 3.4). In case of even k the fixed points of the action are such hyperplanes that are realifications of the $(k/2)$-dimensional complex subspaces of the bundle η. Applying the localization property we obtain $\tau(p_1) \sim i \circ \tau(p_2)$.

Corollary 7.1. *If the vector bundle η is stably isomorphic to the complex vector bundle, then $p_{(2k+1)/2}(\eta) = 0$.*

Proof. Let the bundle $\eta \oplus [N]$ admit the structure of the complex vector bundle. Then

$$p_{(2k+1)/2}(\eta) = p_{(2k+1)/2}(\eta \oplus [N]) = 0.$$

For the theory of complex self-conjugate cobordisms $SC^*(\cdot)$ the following canonical transformations of the cogomology theories exist

$$\mu^{CO}_{SC}\colon CO^*(\cdot) \to SC^*(\cdot) \qquad \text{and} \qquad \mu^{Sp}_{SC}\colon Sp^*(\cdot) \to SC^*(\cdot)$$

where $Sp^*(\cdot)$ is the symplectic cobordism theory (see [6]).

Corollary 7.2. *If the vector bundle η is stably isomorphic to the complex vector bundle, the image $p_{2k/2}(\eta)$ in $SC^*(B)$ under the transformation μ^{CO}_{Sp} belongs to $\operatorname{Im} \mu^{Sp}_{SC}\big(Sp^*(B)\big) \subset SC^*(B)$.*

Proof. Since Pontrjagin classes are stable, without loss of generality we can assume that the bundle η admits a complex structure. Let $\tau(p_1)$ be the transfer map of the bundle $RG^{2n}_{2k}(\eta)$ and $\tau(p_2)$ the transfer map of the bundle $CG^n_k(\eta)$. Denote by $i\colon CG^n_k \to RG^{2n}_{2k}$ the canonical embedding. From Theorem 7.1 we obtain

$$\begin{aligned}
\mu^{CO}_{SC}\big(p_{2k/2}(\eta)\big) &= \mu^{CO}_{SC}\tau(p_1)^*\chi\big(\mathbb{C} \otimes \xi(2k)\big) \\
&= \mu^{CO}_{SC}\tau(p_2)^* i^*\chi\big(\mathbb{C} \otimes \xi(2k)\big) = \mu^{CO}_{SC}\tau(p_2)^*\chi\big(\mathbb{C} \otimes \eta(k)\big) \\
&= \tau(p_2)^*\mu^{CO}_{SC}\chi\big(\mathbb{C} \otimes \eta(k)\big) = \tau(p_2)^*\chi\big(\eta(k) \oplus \overline{\eta(k)}\,\big) \qquad (7.1)
\end{aligned}$$

where $\xi(2k)$ is the $2k$-dimensional tautological vector bundle over $RG^{2n}_{2k}(\eta)$, and $\eta(k)$ is the k-dimensional tautological vector bundle over $CG^n_k(\eta)$. Remark that for any complex vector bundle η the bundle $\eta \oplus \bar{\eta}$ admits the canonical symplectic structure. Therefore, the right side of (7.1) belongs to $\operatorname{Im} \mu^{Sp}_{SC}\big(Sp^*(B)\big)$.

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