TANGENTIAL STRUCTURES ON TORIC MANIFOLDS, AND CONNECTED SUMS OF POLYTOPES

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Abstract. We extend work of Davis and Januszkiewicz by considering *omnioriented* toric manifolds, whose canonical codimension-2 submanifolds are independently oriented. We show that each omniorientation induces a canonical stably complex structure, which is respected by the torus action and so defines an element of an equivariant cobordism ring. As an application, we compute the complex bordism groups and cobordism ring of an arbitrary omnioriented toric manifold. We consider a family of examples $B_{i,j}$, which are toric manifolds over products of simplices, and verify that their natural stably complex structure is induced by an omniorientation. Studying connected sums of products of the $B_{i,j}$ allows us to deduce that every complex cobordism class of dimension > 2 contains a toric manifold, necessarily connected, and so provides a positive answer to the toric analogue of Hirzebruch's famous question for algebraic varieties. In previous work, we dealt only with disjoint unions, and ignored the relationship between the stably complex structure and the action of the torus. In passing, we introduce a notion of connected sum $\#$ for simple n-dimensional polytopes; when P^n is a product of simplices, we describe $P^n \# Q^n$ by applying an appropriate sequence of *pruning operators*, or hyperplane cuts, to Q^n.

1. Introduction

The study of toric varieties (or torus embeddings, as they were originally known) has entranced algebraic geometers since the 1970s, and provides a host of elegant and illuminating examples. Several comprehensive textbooks are now available, by authors such as Ewald [9], Fulton [10], and Oda [15]. In their pioneering paper of 1991 [8], Davis and Januszkiewicz defined the related notion of toric manifold, thereby extending the audience for the fascinating interplay between combinatorics, geometry and topology which characterises the subject.

A toric manifold M^{2n} admits a smooth action of the torus T^n which may be identified locally with the standard action of T^n on $\mathbb{C}^n$; the quotient space is required to be an n-dimensional ball, invested with the combinatorial structure of a simple convex polytope by the fixed point sets of appropriate subtori. A classic example is provided by the complex projective space CP^n, whose quotient polytope is the n-simplex Δ^n. More general examples may fail to be complex, as shown by the connected sum $CP^n \# CP^n$ whose quotient polytope is the product of simplices $\Delta^1 \times \Delta^{n-1}$. However, it is clear from the work of Davis and Januskiewicz that the action of the torus gives rise to a family of

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complex structures on the *stable* tangent bundle; our basic aim is to classify the members of this family in terms of omniorientations, and to develop the consequences for complex cobordism theory.

Our programme has its origins in [5], where we constructed a sequence of stably complex toric manifolds $B_{i,j}$. Although we ignored any relationship between their stably complex structure and the action of the torus, we did confirm that the resulting cobordism classes are multiplicative generators of the complex cobordism ring Ω_*^U. It follows that every cobordism class is represented by a disjoint union of toric manifolds, which are suitably oriented products of the $B_{i,j}$. This situation compares with Hirzebruch's corresponding result [12] for algebraic varieties, and leads to the same question; can the representatives be chosen to be connected? For varieties it remains unanswered, but we prove:

Theorem 6.11. *In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

During our proof we develop the notion of connected sum # for simple polytopes. In the case of the examples $B_{i,j}$ the quotient polytopes are products of simplices; any such P^n determines a sequence of *pruning operators*, whose application to an arbitrary simple polytope Q^n provides an alternative description for $P^n \# Q^n$ in terms of hyperplane cuts.

Davis and Januszkiewicz succeeded in computing the integral homology and cohomology of an arbitrary toric manifold M^{2n}, and our analysis of stably complex structures allows us to extend their computations to complex cobordism theory. We summarise this development in a separate section, having studied the specific case of bounded flag manifolds in [4]. Our statement of results for M^{2n} depends on the choice of stably complex structure, and needs care to make precise. Nevertheless, knowledge of $\Omega_U^*(M^{2n})$ leads to the description of $E^*(M^{2n})$ for any complex oriented cohomology theory, and has already been translated by Strickland [20] into the more sophisticated language of formal groups. Non-oriented theories are are more difficult to deal with, but Bahri and Bendersky have made considerable progress in the case of KO-theory [1].

Recently, Hattori [11] and Masuda [14] have studied torus actions on stably complex manifolds as a generalisation of toric varieties, and our work confirms that omnioriented toric manifolds fall within their framework. Also, Panov [16], [17] has incorporated our notion of omniorientation into his combinatorial description of certain cobordism invariants. Although the methods and objectives of these authors differ significantly from ours, it will be of interest to combine the various approaches in future.

Throughout our work we write T^n for the n-dimensional torus, and refer to its representation by diagonal matrices in $U(n)$ as the *standard* action on $\mathbb{C}^n$. The quotient space of this action is the positive cone

$$\{(x_1, \ldots, x_n) : x_r \geq 0, \text{ for } 1 \leq r \leq n\}$$

in $\mathbb{R}^n$, which we write as $\mathbb{R}^n_{\geqslant}$. We may recover $\mathbb{C}^n$ from the cone as the identification space $(T^n \times \mathbb{R}^n_{\geqslant})/\approx$, where $(t, x) \approx (u, x)$ whenever the coordinates of t differ from the coordinates of u only in those positions r where x_r is zero, and the standard action is induced by multiplication in T^n. We let $\mathbb{R}^n_>$ denote the subspace of vectors whose coordinates are strictly positive. On several occasions we consider smooth manifolds which are locally diffeomorphic to $\mathbb{R}^n_\geqslant$; following the original definitions of the 60s [13], we refer to these as n-dimensional manifolds *with corners*.

We often abbreviate singleton sets such as $\{v\}$ by omitting the brackets.

It is a pleasure to acknowledge the insight we have gained from discussions with several of our colleagues during the preparation of this work, and to nominate Tony Bahri, Yusuf Civan, Taras Panov and Neil Strickland for particular mention. The second author also wishes to apologise for the long delay in completing the manuscript; this was beyond his control, but has led to some confusion over the status of our results.

2. Toric manifolds

We begin with a summary of Davis and Januszkiewicz's treatment of toric manifolds, giving our own interpretation as required for the study of stably complex structures in later sections.

We consider an unordered set $\mathcal{H}$ of m closed halfspaces H in a real affine space $\mathbb{A}^n$. We assume that $m > n$, and that the bounding hyperplanes are in general position, so that no $n{+}1$ of them meet; we also insist that the removal of any halfspace will enlarge the intersection $\cap_{\mathcal{H}} H$. We refer to $\cap_{\mathcal{H}} H$ as a *simple n-polyhedron* P^n, and to each of its intersections with a bounding hyperplane as a *facet*. Once an orthonormal coordinate system is chosen for $\mathbb{A}^n$ we may represent P^n by a matrix inequality $A_P x \geq b$, where A_P is an $\mathcal{H} \times n$ real matrix, and x and b are column vectors in $\mathbb{R}^n$ and $\mathbb{R}^{\mathcal{H}}$ respectively; the rows of A_P are indexed by the elements of $\mathcal{H}$, so that each H is described by the corresponding row of the inequality. We reserve the term *polytope* for a bounded polyhedron.

The standard octahedron, for example, is not simple, because 4-tuples of bounding hyperplanes meet at four of its vertices in $\mathbb{R}^3$.

A simple polyhedron P^n is uniquely determined by its set of facets $\mathcal{F}(P)$, which we abbreviate to $\mathcal{F}$ whenever possible. Given $0 < k \leq n$, every nonempty intersection of k facets forms a *face* of P^n, which has codimension k by general position; conversely, every codimension-k face G determines a unique set $\mathcal{F}_G$ of k facets. Any such face is itself a simple polyhedron G^{n-k}, defined by those facets of P^n which intersect it properly. We may therefore partition $\mathcal{F}$ as

$$\mathcal{F}(G) \cup \mathcal{F}_G \cup \mathcal{D}_G, \tag{2.1}$$

where $\mathcal{D}_G$ consists of the facets disjoint from G. In particular, every vertex v of P^n is determined by a unique set $\mathcal{F}_v$ of n facets, and so lies in a neighbourhood which is linearly isomorphic to the cone $\mathbb{R}^n_\geqslant$. It follows that P^n is an n-dimensional manifold with corners, and has an atlas with one affine chart U_v for each vertex v. Clearly P^n is a convex submanifold of the ambient $\mathbb{A}^n$.

Geometers originally studied polyhedra up to affine equivalence, but the weaker notion of *combinatorial equivalence*, determined by the lattice of faces $\mathfrak{L}_F(P)$, is now equally fashionable; polyhedra are combinatorially equivalent if and only if they are diffeomorphic as manifolds with corners. Many fascinating details, and a host of further references, are given in Ziegler's book [23].

To establish our notation we describe two fundamental families of polytopes in some detail, namely simplices and cubes.

The standard n-simplex Δ^n lies in $\mathbb{R}^n$, and has defining halfspaces

$$(2.2)\quad H_r = \{x : x_r \geq 0\} \text{ for } 1 \leq r \leq n, \text{ and } H_{n+1} = \{x : x_1 + \cdots + x_n \leq 1\},$$

with corresponding facets $D_r = \Delta^n \cap H_r$. Each D_r is a copy of the $(n-1)$-simplex Δ^{n-1} for $1 \leq r \leq n$, whilst D_{n+1} is affinely equivalent to Δ^{n-1}. The codimension k faces D_S are $(n-k)$-simplices, indexed by the k-element subsets S of $\{1, \dots, n+1\}$, and the face lattice $\mathfrak{L}_F(\Delta^n)$ is therefore Boolean of rank n.

The standard n-cube I^n also lies in $\mathbb{R}^n$, and has defining halfspaces

$$(2.3)\qquad H_r^0 = \{x : x_r \geq 0\} \quad \text{and} \quad H_r^1 = \{x : x_r \leq 1\}$$

for $1 \leq r \leq n$, with corresponding facets $C_r^\varepsilon = I^n \cap H_r^\varepsilon$, where $\varepsilon = 0$ or 1. Each C_r^ε is an $(n-1)$-cube I^{n-1}, for $1 \leq r \leq n$. The codimension k faces are $(n-k)$-cubes, indexed by the cartesian coordinates of their centres; these are ternary sequences ξ of length n on $\{\frac{1}{2}, 0, 1\}$, in which $\frac{1}{2}$ occurs $n-k$ times. Thus C_r^ϵ is indexed by $\xi_j = \frac{1}{2}$ for $j \neq r$ and $\xi_r = \epsilon$, whilst the vertices are given by their coordinate sequences of 0s and 1s. The face lattice $\mathfrak{L}_F(I^n)$ has 3^n elements, and is of independent interest to combinatorialists.

We often use the product polytope $I^m \times \Delta^n$. This has facets $C_r^\varepsilon \times \Delta^n$ and $I^m \times D_s$ for $1 \leq r \leq m$ and $1 \leq s \leq n$, written E_r^ε and E_s respectively.

We consider $2n$-dimensional manifolds M^{2n} which are equipped with an action α of the torus T^n, and suppose for convenience that both M^{2n} and α are smooth. Given $t \in T^n$ and $x \in M^{2n}$, we abbreviate $\alpha(t, x)$ to $t \cdot x$ wherever possible. We assume that α is locally equivalent to the standard action $\mathbb{C}^n$, by insisting that every point x of M^{2n} lies in some neighbourhood V, closed under the action of α, for which there is a *θ_x-equivariant* diffeomorphism $h \colon V \to \mathbb{C}^n$; in other words,

$$(2.4)\qquad h(t \cdot y) = \theta_x(t) \cdot h(y)$$

for some automorphism θ_x of T^n, and all $t \in T^n$ and $y \in V$. Given a simple n-polyhedron P^n, we describe M^{2n} as a *toric manifold over* P^n whenever there exists a smooth projection $\pi \colon M^{2n} \to P^n$ whose fibres are the orbits of α. We may display this information as the quadruple $(M^{2n}, \alpha, \pi, P^n)$, and refer to P^n as the *base polyhedron*.

It is customary to insist that M^{2n} and P^n should be compact, but the extra generality will prove helpful in Sections 3 and 6 below.

Each face of P^n of codimension k is the image under π of the fixed point set of some k-dimensional subtorus, for all $0 \leq k \leq n$; for example, the vertices are the image of the fixed points, and the boundary ∂P^n is the image of the points on which T^n fails to act freely. The maps h descend to local diffeomorphisms between P^n and the cone $\mathbb{R}^n_{\geqslant}$, yielding charts for P^n as a manifold with corners. In particular, the charts based on open subsets $U_v \subset P^n$ correspond to a finite T^n-invariant atlas for M^{2n}, each of whose open sets V_x contains a single fixed point $x = \pi^{-1}(v)$. It follows from [8] that π admits smooth right inverses $P^n \to M^{2n}$, from which we select a *preferred section* s, transverse to the orbits. Any other choice differs from s by some map $P^n \to T^n$, and is therefore homotopic to s through right inverses because P^n is contractible. We note that P^n and the quotient M^{2n}/T^n are diffeomorphic as manifolds with corners.

Every facet F of P^n determines a subspace $\pi^{-1}(F)$, readily seen to be a submanifold $X(F)^{2(n-1)} \subset M^{2n}$ with isotropy subgroup a circle $T(F)$ in T^n. As F ranges over $\mathcal{F}_G$ for some face G of codimension k, the $X(F)^{2(n-1)}$ intersect transversally in a submanifold $X(G)^{2(n-k)}$, whose isotropy subgroup $T(G)$ is a k-dimensional subtorus, and is generated by the circles $T(F)$. We therefore have a *characteristic map* $\lambda\colon \mathfrak{L}_F(P) \to \mathfrak{L}_S(T^n)$ into the lattice of subtori of T^n, which preserves the corresponding concept of rank. In this way, we associate the *characteristic pair* (P^n, λ) to $(M^{2n}, \alpha, \pi, P^n)$.

Now let us reverse this process by starting with a pair (P^n, λ), where λ is a rank-preserving map of the lattices above. Note that each point q of ∂P^n lies in the relative interior of a unique face $G(q)$. We use this data to construct the identification space

$$(T^n \times P^n)/\sim\,, \tag{2.5}$$

where $(t, q) \sim (u, q)$ if and only if tu^{-1} lies in the subtorus $\lambda(G(q))$ of T^n. Multiplication on the first coordinate defines an action of T^n on the resulting space, with quotient P^n. Whenever q lies in the interior of P^n the equivalence classes (t, q) are singletons, and have trivial isotropy subgroups; at the other extreme, the fixed points consist of the equivalence classes $(1, v)$, where v ranges over the vertices of P^n. Just as P^n is covered by the open sets U_v, based on the vertices and diffeomorphic to $\mathbb{R}^n_\geqslant$, so the identification space is covered by open sets $(T^n \times U_v)/\sim$, centred on the fixed points $(1, v)$ and homeomorphic to $(T^n \times \mathbb{R}^n_\geqslant)/\approx$, and therefore to $\mathbb{C}^n$. With this structure, the rank-preserving properties of λ ensure that the identification space is a toric manifold over P^n, which we say is *derived from* (P^n, λ).

Given two toric manifolds over the same polyhedron P^n, we deem them to be indestinguishable whenever they are linked by some θ-equivariant diffeomorphism (in the sense of (2.4)) which covers the identity map on P^n; here θ is an automorphism of the torus T^n, and so induces an automorphism θ_* of the lattice $\mathfrak{L}_S(T^n)$. Any such diffeomorphism descends to a *θ-translation* of characteristic pairs, in which the two charactersitic maps differ by θ_*. When θ is the identity, these concepts reduce to equivariant diffeomorphism of toric manifolds and equality of characteristic pairs, respectively. Two θ-equivariant diffeomorphisms f and f' are *equivalent* whenever there exist equivariant diffeomorphisms h_1 and h_2 such that $f \cdot h_1 = h_2 \cdot f'$.

Proposition 2.6. *For any automorphism θ, the assignment of characteristic pairs defines a bijection between equivalence classes of θ-equivariant diffeomorphisms of toric manifolds, and θ-translations of pairs (P^n, λ).*

Proof. To prove bijectivity, we show that the inverse assignment is given by taking derived toric manifolds; to each θ-translation $(P^n, \lambda) \to (P^n, \theta_*(\lambda)$ we associate the θ-equivariant diffeomorphism $\theta \times 1\colon (T^n \times P^n)/\sim\, \to (T^n \times P^n)/\sim_\theta$, where $(t, q) \sim_\theta (u, q)$ if and only if $tu^{-1} \in \theta_*(\lambda)(G(q))$.

It follows directly from the definitions that $\theta \times 1$ descends to the original θ-translation $(P^n, \lambda) \to (P^n, \theta_*(\lambda)$ of characteristic pairs. If, on the other hand, we start with a θ-equivariant diffeomorphism $f\colon M_1^{2n} \to M_2^{2n}$ (or its equivalent), then $\theta\times 1$ is derived from the corresponding θ-translation of characteristic pairs. But the preferred section s_1 for M_1^{2n} automatically extends to an equivariant diffeomorphism $S_1\colon (T^n\times P^n)/{\sim} \to M_1^{2n}$, and the section $s_2 = f\cdot s_1$ extends to an eqivariant diffeomorphism $S_2\colon (T^n\times P^n)/{\sim_\theta} \to M_2^{2n}$; thus $f\cdot S_1 = S_2\cdot(\theta\times 1)$, whence f and $\theta\times 1$ are equivalent, as required. $\square$

In subsequent sections it will be convenient to replace $(M^{2n}, \alpha, \pi, P^n)$ by its *derived form* (2.5), and use S to transfer our constructions back to M^{2n}. We abbreviate $(T^n\times P^n)/{\sim}$ to $M_\bullet^{2n}$.

The *facial submanifolds* $X(G)^{2(n-k)}$ are central to cobordism calculations, and form a lattice $\mathfrak{L}_X(M^{2n})$ which is isomorphic to $\mathfrak{L}_F(P)$. We write $\nu(G)$ for the normal $2k$-bundle of the embedding $X(G)^{2(n-k)} \subset M^{2n}$. We may assume that $T(G)$ acts on the fibres of $\nu(G)$ isometrically with respect to a T^n-invariant metric; the transformations acting tangentially form an $(n-k)$-dimensional subtorus $T^\top(G)$, which splits T^n as $T(G)\times T^\top(G)$ and invests $X(G)^{2(n-k)}$ with its own toric structure. We refer to this action as the *restriction* of α, and note that different choices of basis for $T^\top(G)$ correspond to θ-equivariantly diffeomorphic versions of $X(G)^{2(n-k)}$. Thus $\mathfrak{L}_X(M^{2n})$ is a lattice of subtoric manifolds.

Under S the facial submanifold $X(G)^{2(n-k)}$ corresponds to the identification subspace $(T^\top(G)\times G^{n-k})/{\sim}$, which we may equate with the derived form $X(G)_\bullet^{2(n-k)}$ once a basis is chosen for $T^\top(G)$.

We are particularly interested in three families of toric manifolds. They all happen to be toric varieties, but admit alternative stably complex structures which arise naturally in the context of complex cobordism theory. In giving their description, we denote a generic element t of T^n by $(t_1,\dots,t_n)$, and for each $1\le r\le n$ we write T_r for the rth coordinate circle, defined by $t_k=1$ unless $k=r$; we write the diagonal circle as T_δ. In each case we leave to readers the task of verifying that the action is locally standard .

Example 2.7. Complex projective space CP^n is a toric manifold with respect to the action induced by $t\cdot z = (t_1z_1,\dots,t_nz_n,z_{n+1})$ on the unit sphere in $\mathbb{C}^{n+1}$, and the projection $\pi([z]) = (|z_1|^2,\dots,|z_n|^2)$ onto the n-simplex Δ^n.

In the notation of (2.2), the facial submanifold $X(D_r)$ is a copy of CP^{n-1}; the normal bundle $\nu(D_r)$ is isomorphic (as real 2-plane bundles) to the Hopf bundle $\zeta(n-1)$, and is the restriction of $\zeta(n)$. The characteristic map is given by $\lambda(T^{D_r}) = T_r$ for $1\le r\le n$, and $\lambda(T^{D_{n+1}}) = T_\delta$. Each $X(D_S)$ is a copy of $CP^{n-|S|}$, and the lattice $\mathfrak{L}_X(CP^n)$ is Boolean of rank n.

The stable tangent bundle arising from the complex algebraic structure is given by a canonical isomorphism $\tau(CP^n)\oplus\mathbb{C}\cong(n+1)\overline{\zeta}(n)$ [19].

Example 2.8. The bounded flag manifold B_n is described in [4], and consists of complete flags U in $\mathbb{C}^{n+1}$ for which U_k contains the subspace $\mathbb{C}^{k-1}$ (spanned by the first $k-1$ standard basis vectors) for $2\le k\le n$; thus U is equivalent to a sequence of lines $L_k < \mathbb{C}_k\oplus L_{k+1}$ for $1\le k\le n$, where $\mathbb{C}_k$ denotes the kth coordinate line, and $L_{n+1}=\mathbb{C}_{n+1}$. Then B_n is a toric manifold with respect to the action induced by $t \cdot z = (t_1 z_1, \dots, t_n z_n, z_{n+1})$ on $\mathbb{C}^{n+1}$, and projection $\pi(U) = (\pi(L_1), \dots, \pi(L_n))$ onto the n-cube I^n, where $\pi(L_k)$ is defined in $\mathbb{R}_{\geqslant}$ by projecting a unit vector onto $\mathbb{C}_k$ and taking the square of its modulus. For each $1 \leq k \leq n$, complex line bundles $\gamma_k(n)$ and $\rho_k(n)$ are defined over B_n by assigning to any bounded flag U the line L_k and the orthogonal complement $L_{k,k+1}$ of L_k in $\mathbb{C}_k \oplus L_{k+1}$ respectively. By convention we set $\gamma_0(n)$ and $\gamma_{n+1}(n)$ to be trivial, and identify $\rho_0(n)$ with $\gamma_1(n)$.

In the notation of (2.3), the facial submanifold $X(C_r^0)$ is a copy of B_{n-1}, whose flags lie in $\mathbb{C}^{\{1,\dots,n+1\}\setminus r}$; the normal bundle $\nu(C_r^0)$ is the restriction of $\gamma_r(n)$. On the other hand, $X(C_r^1)$ is a copy of $B_{r-1} \times B_{n-r}$, where the flags of the factors lie in $\mathbb{C}^r$ and $\mathbb{C}^{\{r+1,\dots,n+1\}}$ respectively; $\nu(C_r^1)$ is the restriction of $\rho_r(n)$. The submanifolds $X(C_r^0)$ and $X(C_r^1)$ are labelled in [4] as $Y_{\{1,\dots,n\}\setminus r}$ and $X_{\{1,\dots,n\}\setminus r}$ respectively. The characteristic map is given by $\lambda(T^{C_r^0}) = T_r$ and $\lambda(T^{C_r^1}) = T_\delta < T^r$, where the latter is embedded in T^n via the first r coordinates, for $1 \leq r \leq n$. The lattice $\mathfrak{L}_X(B_n)$ is isomorphic to $\mathfrak{L}_F(I^n)$.

Each B_n is the sphere bundle of $\gamma_1 \oplus \mathbb{R}$ over B_{n-1}. As detailed in [18], this leads to a stably complex structure $\tau(B_n) \oplus \mathbb{R}^2 \cong \oplus_{k=2}^{n+1} \gamma_k(n) \oplus \mathbb{C}$, which plays an important rôle in complex cobordism theory despite bounding the associated disk bundle.

Example 2.9. The manifold $B_{i,j}$ (for integers $0 \leq i \leq j$) is introduced in [5], and consists of pairs (U, W), where U is a bounded flag in $\mathbb{C}^{i+1}$ and W is a line in $U_1^\perp \oplus \mathbb{C}^{j-i}$. So $B_{i,j}$ is a smooth CP^{j-1}-bundle over B_i. It has dimension $2(i+j-1)$, and is a toric manifold with respect to the action induced by

$$t \cdot (z, w) = (t_1 z_1, \dots, t_i z_i, z_{i+1}, t_{i+1} w_1, \dots, t_{i+j-1} w_{j-1}, w_j)$$

on $\mathbb{C}^{i+1} \times (U_1^\perp \oplus \mathbb{C}^{j-i})$, where the coordinates of w are chosen with respect to the decomposition $L_{1,2} \oplus L_{2,3} \oplus \cdots \oplus L_{i,i+1} \oplus \mathbb{C}^{\{i+2,\dots,j+1\}}$. Projection onto the product $I^i \times \Delta^{j-1}$ is defined as $\pi(U, W) = (\pi(U), \pi(W))$, by combining Examples 2.7 and 2.8. For each $1 \leq k \leq i$, complex line bundles $\gamma_k(i)$ and $\rho_k(i)$ are defined over $B_{i,j}$ by pullback from B_i; similarly, ζ is defined by considering W as a line in $\mathbb{C}^{j+1}$.

The facial submanifolds $X(E_r^0)$ and $X(E_s)$ are copies of $B_{i-1,j}$ and $B_{i,j-1}$ respectively, for all $1 \leq r \leq i$ and $i+1 \leq s \leq j$; the corresponding normal bundles $\nu(E_r^0)$ and $\nu(E_s)$ are the restrictions of $\gamma_r(i)$ and ζ. The manifolds $X(E_r^1)$ and $X(E_s)$ for $1 \leq s \leq i$ are new. The characteristic map is given by $\lambda(T^{C_r^0}) = \{(t, t^{-1})\}$ in $T_r \times T_{i+r}$, and $\lambda(T^{C_r^1}) = \{(t, \dots, t, t^{-1}, \dots, t^{-1})\}$ in $T^r \times T^{r-1}$, where T^r and T^{r-1} are embedded in T^{i+j-1} by the first r and the $\{i+1, \dots, i+r-1\}$ coordinates respectively; and also by $\lambda(T^{D_s}) = T_{i+s}$ for $1 \leq s \leq j-1$ and $\lambda(T^{D_j}) = T_\delta < T^{j-1}$, where T^{j-1} is embedded in T^{i+j-1} as the last $j-1$ coordinates. The lattice $\mathfrak{L}_X(B_{i,j})$ is isomorphic to the product $\mathfrak{L}_F(I^i) \times \mathfrak{L}_F(\Delta^{j-1})$.

The projection onto B_i and the classifying map of ζ together provide a smooth embedding of $B_{i,j}$ in $B_i \times CP^j$, whose normal bundle is $\gamma_1(i) \otimes \zeta$. Combining this with the bounding structure on B_i and the varietal structure on CP^j yields the isomorphism $\tau(B_{i,j}) \oplus (\gamma_1(i) \otimes \overline{\zeta}) \oplus \mathbb{C} \cong \oplus_{k=2}^{i+1} \gamma_k(i) \oplus (j+1)\overline{\zeta}$ which defines the stably complex structure used implicitly in [5].

Further examples are provided by taking connected sums of the above, as outlined in [8]; however, the resulting tangent bundles are rarely complex. As we now explain, the best we can generally expect is a complex structure on the *stable* tangent bundle.

3. Stably complex structure

In order to describe our stably complex structures with appropriate precision, we need to assign a collection of orientations to each toric manifold. Davis and Januszkiewicz [8] sometimes incorporate equivalent information into their notion of characteristic map, but they do so implicitly, and without considering the dependence of the resulting structures on their choice.

Given any toric manifold $(M^{2n}, \alpha, \pi, P^n)$ and any facet F of P^n, the action of $T(F)$ allows us to interpret the 2-plane bundle $\nu(F)$ as a complex line bundle. Two complex structures are possible, which differ by conjugation and correspond to opposite orientations. An *omniorientation* of $(M^{2n}, \alpha, \pi, P^n)$ consists of a choice of such orientation for every facet F; there are therefore 2^m omniorientations in all, each of which is preserved by α because T^n is connected. By transversality, an omniorientation determines an orientation (and also a complex structure) for $\nu(G)$, given any face G of P^n.

By analogy, we refer to a characteristic pair (P^n, λ) as *directed* if the circle $\lambda(F)$ is oriented for every facet F of P^n. We may then replace the lattice map λ by an epimorphism $\ell\colon T^{\mathcal{F}} \to T^n$, which encodes each of the isomorphisms $T \to T(F)$ determined by the orientation of the latter. We label ℓ a *directed* characteristic map, or *dicharacteristic*, and write (P^n, ℓ) for a directed characteristic pair; each λ is represented by 2^m distinct dicharacteristics.

The complex structures implicit in [8] need careful interpretation precisely because λ is used there to denote both characteristic and dicharacteristic maps.

The characteristic pair of an omnioriented toric manifold is obviously directed, and the toric manifold derived from a directed characteristic pair is omnioriented. For any automorphism θ of T^n, we insist that a θ-equivariant diffeomorphism between omnioriented toric manifolds should respect each of the 2^m facial orientations; correspondingly, a θ-translation of directed pairs must satisfy $\ell_2 = \theta \cdot \ell_1$. In this context, the following extension of Proposition 2.6 is immediate.

Proposition 3.1. *The assignment of directed characteristic pairs defines a bijection between equivalence classes of θ-equivariant diffeomorphisms of omnioriented toric manifolds, and θ-translations of pairs (P^n, ℓ).*

Transversality ensures that an omniorientation of M^{2n} restricts to an omniorientation of any facial submanifold $X(G)^{2(n-k)}$. If the former corresponds to the dicharacteristic ℓ under Proposition 3.1, the latter corresponds to its restriction

$$\ell_{X(G)}\colon T^{\mathcal{F}(G)} \to T^{\top}(G) \tag{3.2}$$

under the partition (2.1) of $\mathcal{F}(P)$.

For each omniorientation of (M^{2n},α,π,P^n) we now construct the induced complex structures on M^{2n}. We focus initially on the base polyhedron, which we assume to be defined in $\mathbb{R}^n$ for convenience.

We recall the presentation of P^n as a matrix inequality, and interpret the $\mathcal{H}\times n$ matrix A_P as a linear transformation $A_P\colon\mathbb{R}^n\to\mathbb{R}^{\mathcal{F}}$. We abbreviate the n-dimensional image $A_P(\mathbb{R}^n)$ to V_P, and write $V_P^\perp$ for its $(m-n)$-dimensional orthogonal complement in $\mathbb{R}^{\mathcal{F}}$ (with respect to the standard inner product). Since the points of P^n are specified by the constraint $A_Px\geq b$, it follows that the intersection of the affine subspace V_P-b with the positive cone $\mathbb{R}^{\mathcal{F}}_{\geqslant}$ is a copy of P^n; it is embedded in $\mathbb{R}^{\mathcal{F}}$ as the space of functions $\{d(p,\ ):\mathcal{F}\to\mathbb{R}_{\geqslant}\}$, where $d(p,F)$ is the euclidean distance between p and the hyperplane defining F for each $p\in P^n$ and each facet F. We refer to this embedding as $d_{\mathcal{F}}$. We sometimes identify P^n with its image $d_{\mathcal{F}}(P^n)$, which actually lies in the subspace

$$W^{\mathbb{R}}(P)=\{f\colon\mathcal{F}\to\mathbb{R}_{\geqslant}\text{ such that }f^{-1}(0)\in\mathfrak{L}_F(P)\}\tag{3.3}$$

of $\mathbb{R}^{\mathcal{F}}_{\geqslant}$.

For any subset $\mathcal{G}\subseteq\mathcal{F}$ of facets, we may realise $\mathbb{R}^{\mathcal{G}}$ as a subspace of $\mathbb{R}^{\mathcal{F}}$ by choosing F-coordinates to be 0 for all F in $\mathcal{F}\setminus\mathcal{G}$. Thus $W^{\mathbb{R}}(P)$ consists of the union of open cones $\bigcup\mathbb{R}^{\mathcal{C}_G}_{>}$, where $\mathcal{C}_G$ denotes the complement of $\mathcal{F}_G$ in $\mathcal{F}$, and G ranges over $\mathfrak{L}_F(P)$; the union is topologised by embedding the cones in $\mathbb{R}^{\mathcal{F}}_{\geqslant}$ in the obvious fashion, so that the interior of G is embedded in $\mathbb{R}^{\mathcal{C}_G}_{>}$ for each face G. Clearly $W^{\mathbb{R}}(P)$ is a noncompact m-dimensional manifold with corners, and never contains the zero vector.

The open cone $\mathbb{R}^{\mathcal{F}}_{>}$ is an abelian topological group under coordinatewise multiplication $*$. It decomposes as $\exp(V_P^\perp)\times\exp(V_P)$ by exponentiating the additive splitting of $\mathbb{R}^{\mathcal{F}}$ as $V_P^\perp\oplus V_P$. The group $\exp(V_P^\perp)$ therefore acts smoothly on $\mathbb{R}^{\mathcal{F}}_{>}$ by $*$, with quotient space $\exp(V_P)$. This action restricts to each embedded cone $\mathbb{R}^{\mathcal{G}}_{>}$, and extends to $\mathbb{R}^{\mathcal{G}}_{\geqslant}$ and $W^{\mathbb{R}}(P)$.

Proposition 3.4. *As manifold with corners, $W^{\mathbb{R}}(P)$ is canonically diffeomorphic to the cartesian product $P^n\times\exp(V_P^\perp)$.*

Proof. Given $p\in P^n$ and $a\in V_P^\perp$, we consider tangents to the orbit $\exp(V_P^\perp)*p$ at $\exp(a)*p$. One such has direction vector $a*\exp(a)*p$, whose inner product with a is given by $\sum_{\mathcal{F}}a_F^2\exp(a_F)p_F$. Since this quantity is strictly positive the orbit meets P^n only at p, (and the intersection is transverse). On the other hand, given any point x in the open cone $\mathbb{R}^{\mathcal{C}_G}_{>}$, the orbit $\exp(V_P^\perp)*x$ meets P^n in an interior point of G, for each face G; this follows by taking logarithms and considering the decomposition of $\mathbb{R}^{\mathcal{G}}$ into $V_G^\perp\oplus V_G$. The required diffeomorphism is therefore given by the map $(p,\exp(a))\mapsto\exp(a)*p$. □

Corollary 3.5. *The embedding of P^n in $\mathbb{R}^{\mathcal{F}}_{\geqslant}$ as manifolds with corners has trivial normal bundle; each choice of basis for $V_P^\perp$ provides a framing.*

Proof. Since exponentiation is a diffeomorphism, $W^{\mathbb{R}}(P)$ is a tubular neighbourhood of the embedding; each choice of basis for $V_P^\perp$ therefore trivialises the normal bundle. □

We consider the identification space $(T^{\mathcal{F}} \times W^{\mathbb{R}}(P))/{\approx}$, denoted by $W(P)$, as a complexified form of the tubular neighbourhood. Such a space has also been introduced by Buchstaber and Panov [3], as an extension of a construction for toric varieties [2], [7]. By (3.3), $W(P)$ embeds in $\mathbb{C}^{\mathcal{F}}$ as the space of complex valued functions whose zero-set is $\mathcal{F}_G$ for some face G of P^n. The multiplicative group $(\mathbb{C}_\times)^{\mathcal{F}}$ of vectors with nonzero coordinates acts on $\mathbb{C}^{\mathcal{F}}$ by $*$, and the subgroups $T^{\mathcal{F}}$ and $\exp(V_P^\perp)$ restrict to $W(P)$ by construction.

We now turn to the omniorientation of $(M^{2n}, \alpha, \pi, P^n)$, and write $K(\ell)$ for the kernel of the dicharacteristic; it is an $(m-n)$-dimensional subtorus of $T^{\mathcal{F}}$, and therefore also acts on $W(P)$ by $*$. The quotient of $T^{\mathcal{F}}$ by $K(\ell)$ is, by definition, the original torus T^n, from which we deduce that the projection

$$(T^{\mathcal{F}} \times W^{\mathbb{R}}(P))/\approx \;\longrightarrow\; (T^n \times P^n)/\sim \tag{3.6}$$

displays $W(P)$ as a smooth, principal $K(\ell) \times \exp(V_P^\perp)$-bundle over the derived form $M_\bullet^{2n}$. We abbreviate $K(\ell) \times \exp(V_P^\perp)$ to $H(\ell)$; since it is a subgroup of $(\mathbb{C}_\times)^{\mathcal{F}}$, the embedding of $W(P)$ in $\mathbb{C}^{\mathcal{F}}$ is $H(\ell)$-equivariant.

The tangent bundle of $W(P)$ inherits a natural complex structure from that of $(\mathbb{C}_\times)^{\mathcal{F}}$, and its quotient by the action of $H(\ell)$ provides our stably complex structure on M^{2n}. The details, however, need care; they involve extending Sczarba's analysis of [22] to (3.6), circumventing his restriction to compact fibres. We obtain a canonical isomorphism

$$\tau(M_\bullet^{2n}) \oplus \tau^{||} \cong \sigma(\mathcal{F}) \tag{3.7}$$

of real $2m$-plane bundles, where $\tau^{||}$ is the quotient of the tangents along the fibres by $H(\ell)$, and $\sigma(\mathcal{F})$ is the $\mathbb{C}^{\mathcal{F}}$-bundle associated to (3.6). We equip each of these bundles with the standard inner product, and insist that $H(\ell)$ acts on the fibres $\mathbb{C}^{\mathcal{F}}$ by projection onto its maximal compact subgroup $K(\ell)$. Of course $\sigma(\mathcal{F})$ is isomorphic to the m-fold sum of complex line bundles $\oplus_{\mathcal{F}}\sigma(F)$, where $\sigma(F)$ has total space $W(P) \times_{H(\ell)} \mathbb{C}^F$.

Theorem 3.8. *An omniorientation of a toric manifold induces a stably complex structure on the derived form; it is defined uniquely up to homotopy.*

Proof. Following (3.7), we must identify $\tau^{||}$ with $\mathbb{R}^{2(m-n)}$. Any choice of basis for the Lie algebra of $H(\ell)$ will have this effect; since the space of bases has two connected components, it actually suffices to give an orientation. But $H(\ell)$ is the kernel of an epimorphism $\mathbb{C}_\times^{\mathcal{F}} \to T^n \times \exp(V_P)$ and $\mathbb{C}^{\mathcal{F}}$ is canonically oriented, so it remains to orient the domain; since the latter is isomorphic to $(\mathbb{C}_\times)^n$, we may simply apply the standard orientation of T^n. The resulting isomorphism

$$\tau(M_\bullet^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \sigma(\mathcal{F}) \tag{3.9}$$

provides the structure we seek. $\square$

If we use the opposite orientation for T^n, and hence for $T^n \times \exp(V_P)$, we obtain a second stably complex structure. This is compatible with the opposite orientation on $M_\bullet^{2n}$, and its complex cobordism class is the negative of that represented by (3.9). We emphasise that the isomorphism (3.9) does not depend on any ordering of $\mathcal{F}$.

Theorem 3.8 gives a global description for any toric manifold, with its induced complex structure, as the quotient of a complex space; given a nonsingular toric variety [7], it yields the stabilisation of the underlying complex structure.

To continue our investigation of induced complex structures, we need a technical lemma. It considers directed characteristic pairs (P_1^n, ℓ_1) and (P_2^n, ℓ_2), with omnioriented derived forms M_1^{2n} and M_2^{2n} respectively, and assumes given closed halfspaces H_1 and H_2 in $\mathbb{R}^n$, and a diffeomorphism $f\colon P_1^n \setminus H_1 \to P_2^n \setminus H_2$ as manifolds with corners. We abbreviate $P_1^n \setminus H_1$ to O^n, and write $\mathcal{C}_1$ and $\mathcal{C}_2$ for the sets of facets contained in H_1 and H_2 respectively. We partition $\mathcal{F}_1$ as $\mathcal{E} \cup \mathcal{C}_1$ and $\mathcal{F}_2$ as $\mathcal{E} \cup \mathcal{C}_2$, where $\mathcal{E}$ consists of those facets which intersect O^n; we use f to identify $\mathcal{E}$ with the set of facets intersecting $f(O^n)$.

Lemma 3.10. *If f preserves dicharacteristics on $\mathcal{E}$, it lifts to a an equivariant diffeomorphism*

$$f^+\colon (T^n \times O^n)/\sim\ \longrightarrow (T^n \times f(O^n))/\sim\ ;$$

f^+ respects the stably complex structures obtained by restriction from those induced on M_1^{2n} and M_2^{2n} by their respective omniorientations.

Proof. The existence of f^+ is assured by the fact that ℓ_1 and ℓ_2 agree on $\mathcal{E}$; we write their common restriction as ℓ.

Our data implies that $d_{\mathcal{F}_1}(O^n)$ maps diffeomorphically onto $d_{\mathcal{E}}(O^n)$ under the projection of $\mathbb{R}^{\mathcal{F}_1}$ onto $\mathbb{R}^{\mathcal{E}}$, so that $\exp(V_{P_1}^\perp)$ is isomorphic to $\exp(V_O^\perp) \times \mathbb{R}_>^{\mathcal{C}_1}$ in $\mathbb{R}_\geqslant^{\mathcal{E}} \times \mathbb{R}_\geqslant^{\mathcal{C}_1}$. Since $W^{\mathbb{R}}(P_1)|_O$ is given by $\exp(V_{P_1}^\perp) * O^n$ in $\mathbb{R}_\geqslant^{\mathcal{F}_1}$, it is equivariantly diffeomorphic to $\exp(V_O^\perp) * O^n \times \mathbb{R}_>^{\mathcal{C}_1}$ with respect to the splitting of $\exp(V_{P_1}^\perp)$; we write $W^{\mathbb{R}}(O)$ for the subspace $\exp(V_O^\perp) * O^n \subset \mathbb{R}_\geqslant^{\mathcal{E}}$. We also note that $K(\ell_1)$ splits as $K(\ell) \times T^{\mathcal{C}_1}$. The stably complex structure on $(T^n \times O^n)/\sim$ given by factoring out the action of $H(\ell_1)$ on $W(P)|_O$ therefore differs from that given by factoring out the action of $H(\ell)$ on $W(O)$ only by the trivial summand $\mathbb{C}^{\mathcal{C}_1}$, so that we may consider the latter as obtained from M_1^{2n} by restriction. Similar remarks apply to $(T^n \times f(O^n))/\sim$ and M_2^{2n}.

To show that f^+ respects the restricted complex structures, we then choose an isomorphism $\exp(V_O^\perp) \to \exp(V_{f(O)}^\perp)$; this immediately extends to an equivariant diffeomorphism $W^{\mathbb{R}}(O) \to W^{\mathbb{R}}(f(O))$, and therefore to an equivariant diffeomorphism $W(O) \to W(f(O))$ which preserves the action of $T^{\mathcal{E}}$. The differential of the second diffeomorphism is complex linear, and reduces to df^+ on the quotient tangent bundles, as required. □

Our first corollary to Lemma 3.10 deals with the reliance of Theorem 3.8 on the hyperplanes defining P^n, in apparent contradiction to the fact that an omniorientation of M^{2n} involves only the action α.

Corollary 3.11. *Given a common omniorientation for $(M^{2n}, \alpha, \pi_1, P_1^n)$ and $(M^{2n}, \alpha, \pi_2, P_2^n)$, the derived forms are equivariantly diffeomorphic; the induced stably complex structure therefore depends only on the combinatorial type of the base polyhedron.*

Proof. The data yields a diffeomorphism $f\colon P_1^n \to P_2^n$, with $f \cdot \pi_1 = \pi_2$; we then apply Lemma 3.10 to f with $O^n = P_1^n$, and to f^{-1} with $O^n = P_2^n$. □

Our second application relates two stably complex structures which are naturally prescribed on $X(G)_\bullet^{2(n-k)}$ by Theorem 3.8, for any codimension k face G. One is induced by the restricted omniorientation associated with the dicharacteristic $\ell_{X(G)}$ of (3.2); the other is the restriction to $X(G)_\bullet^{2(n-k)}$ of the structure induced on $M_\bullet^{2n}$, using the complex structure given on $\nu(G)$ by the omniorientation.

We confirm that these are equivalent in Theorem 3.13. The proof involves an auxiliary polyhedron R^n, which is defined by expressing G^{n-k} as an intersection of halfspaces in $\mathbb{R}^{n-k}$ and taking products with $\mathbb{R}_\geqslant^{\mathcal{F}_G}$; the result is a simple n-polyhedron in $\mathbb{R}^{n-k}\times\mathbb{R}^{\mathcal{F}_G}$, whose facets $\mathcal{F}(R)$ may be partitioned as $\mathcal{F}(G)\cup\mathcal{F}_G$. Then we have

$$d_{\mathcal{F}(R)}(R^n) = d_{\mathcal{F}(G)}(G^{n-k}) \times \mathbb{R}_\geqslant^{\mathcal{F}_G} \quad \text{in} \quad \mathbb{R}_\geqslant^{\mathcal{F}(R)}. \tag{3.12}$$

The restriction of ℓ to $\mathcal{F}(R)$ agrees with $\ell_{X(G)}$ on $\mathcal{F}(G)$ and defines an omnioriented toric manifold L^{2n} over R^n. We invest L^{2n} with the induced stably complex structure, and note that $X(G)_\bullet^{2(n-k)}$ is a facial submanifold, to which the omniorientations of $M_\bullet^{2n}$ and L^{2n} have common restriction.

Theorem 3.13. *The two stably complex structures on $X(G)_\bullet^{2(n-k)}$ are homotopic; that is, restriction and induction commute for $M_\bullet^{2n}$.*

Proof. Considering G as a face of R^n, we note that $W^{\mathbb{R}}(R) = W^{\mathbb{R}}(G) \times \mathbb{R}_\geqslant^{\mathcal{F}_G}$ in $\mathbb{R}_\geqslant^{\mathcal{F}(R)}$, that $K(\ell_L) = K(\ell_{X(G)_\bullet}) \times 1$ in $T^{\mathcal{F}(G)} \times T^{\mathcal{F}_G}$, and that $V_R^\perp = V_G^\perp \times 0$ in $\mathbb{R}^{\mathcal{F}(G)} \times \mathbb{R}^{\mathcal{F}_G}$ by (3.12). Thus $W(R) = W(G) \times \mathbb{C}^{\mathcal{F}_G}$, and $H(\ell_L)$ acts as $H(\ell_{X(G)_\bullet}) \times 1$. It follows that the normal bundle ν of $X(G)_\bullet^{2(n-k)}$ in L^{2n} has total space $W(G) \times_{H(\ell_L)} \mathbb{C}^{\mathcal{F}_G}$, and therefore that restriction and induction commute for L^{2n}.

Considering G as a face of P^n, we may no longer appeal to (3.12). Instead, we apply Lemma 3.10 with $P_1^n = R^n$ and $P_2^n = P^n$; we let H_2 have complement an open tubular neighbourhood $N(G)$ of G, and $f\colon R^n \to N(G)$ be any diffeomorphism extending the identity on G. The Lemma provides an equivariant diffeomorphism f^+ between L^{2n} and an open tubular neighbourhood of $X(G)_\bullet^{2(n-k)}$ in $M_\bullet^{2n}$. By construction, f^+ is compatible with the stable complex structures induced by ℓ_L and ℓ respectively, and defines an isomorphism between ν and the normal bundle $\nu_\bullet(G)$ of $X(G)_\bullet^{2(n-k)}$ in $M_\bullet^{2n}$. This isomorphism is well-defined up to homotopy, and therefore confirms that restriction and induction commute for $M_\bullet^{2n}$. □

Before pulling our constructions back to M^{2n} we consider how the bundles $\sigma(F)$ restrict to $X(G)_\bullet^{2(n-k)}$, for any facet F and any face G of codimension k.

Proposition 3.14. *For any facet D disjoint from G, the restriction $\sigma(D)|_{X(G)_\bullet}$ is trivial; on the other hand, $\sigma(\mathcal{F}_G)|_{X(G)_\bullet}$ is isomorphic to $\nu_\bullet(G)$.*

Proof. The first statement follows from the proof of Lemma 3.10 by choosing $P_1^n = P^n$, and letting O^n be a tubular neighbourhood of G; then D lies in $\mathcal{C}_1$, and $\sigma(D)|_{X(G)_\bullet}$ is the corresponding coordinate line bundle in the trivial summand $\mathbb{C}^{\mathcal{C}_1}$. For the second statement, we note that the proof of Theorem 3.13 identifies the total space of $\nu_\bullet(G)$ with $W(P)|_{X(G)_\bullet} \times_{H(\ell)} \mathbb{C}^{\mathcal{F}_G}$. □

Proposition 3.14 leads to an alternative description of $(S^{-1})^*\sigma(F)$, which simplifies subsequent calculations in cobordism theory. We express the orientation of $\nu(F)$ as an integral Thom class in the cohomology group $H^2(M(\nu(F)))$, represented by a complex line bundle over the Thom complex $M(\nu(F))$. We pull this back along the Pontryagin-Thom collapse $M^{2n} \to M(\nu(F))$, and label the resulting bundle $\rho(F)$.

Lemma 3.15. *The line bundles $\sigma(F)$ and $S^*\rho(F)$ are isomorphic over $M_\bullet^{2n}$.*

Proof. Since S restricts to a preferred section for $X(F)^{2(n-1)}$, so S^{-1} pulls $\nu_\bullet(F)$ back to $\nu(F)$. From Proposition 3.14, we deduce that $(S^{-1})^*\sigma(F)$ is isomorphic to $\nu(F)$ over $X(F)^{2(n-1)}$, and is trivial over the complement. But these properties characterise $\rho(F)$. □

We refer to the $\rho(F)$ as the *facial bundles* of M^{2n}, to distinguish them from the canonical line bundles L_F of algebraic geometry, defined when M^{2n} is also a toric variety. In fact $\rho(F)$ and L_F are either isomorphic or complex conjugate.

Theorem 3.16. *An omniorientation of a toric manifold $(M^{2n}, \alpha, \pi, P^n)$ induces a canonical stably complex structure on M^{2n}, which is preserved by the action α; for each facial submanifold $X(G)^{2(n-k)}$, the restriction of this structure is homotopic to that induced by the restricted omniorientation.*

Proof. Pulling (3.9) back along S^{-1} yields the complex structure on M^{2n}, which Lemma 3.15 converts to an isomorphism $\tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \rho(\mathcal{F})$. Different choices of preferred section s yield homotopic isomorphisms, and therefore homotopic stably complex structures, because the corresponding diffeomorphisms S are isotopic. By Proposition 3.14, the complex structure on $\nu(G)$ is given by an isomorphism $\nu(G) \cong \rho(\mathcal{F}_G)$, which is defined uniquely up to homotopy. The structures restrict to $X(G)^{2(n-k)}$ as claimed, by appeal to Theorem 3.13. □

Theorem 3.16 allows us to compute the complex bordism and cobordism of an arbitrary toric manifold, as explained in Section 5 below. It also shows that any choice of omniorientation leads to an *equivariant* complex cobordism class, as defined, for example, in Comezaña [6]. Equivariant complex cobordism is currently under active development, and we expect the rôle played by toric manifolds to be clarified in future work.

4. Examples

We consolidate our results by returning to the examples of Section 2. It is convenient to follow Davis and Januszkiewicz by simplifying the dicharacteristic to a function which assigns to each facet F of P^n a primitive vector $\ell(F)$ in $\mathbb{Z}^n$; this is obtained by applying the induced map $d\ell$ of Lie algebras to the positively oriented unit tangent vector of T^F.

Example 4.1. For CP^n as in Example 2.7, we note that $W^{\mathbb{R}}(\Delta^n)$ is $\mathbb{R}^{n+1}_{\geqslant} \setminus 0$, and therefore that $W(\Delta^n)$ is isomorphic to $\mathbb{C}^{n+1} \setminus 0$, by ordering the facets of Δ^n as in (2.2). The dicharacteristic chosen by Davis and Januszkiewicz, albeit implicitly, is

$$\ell'(D_r) = \begin{cases} (0, \ldots, 0, 1, 0, \ldots, 0) & \text{for } 1 \leq r \leq n \\ (1, 1, \ldots, 1) & \text{for } r = n+1, \end{cases} \tag{4.2}$$

so that $K(\ell')$ is the subcircle $\{(t, \ldots, t, t^{-1})\}$ of T^{n+1}. Since the normal vector to Δ^n in $\mathbb{R}^{n+1}_{\geqslant}$ is $(1, \ldots, 1, n^{1/2})$, the action of $H(\ell')$ on $W(\Delta^n)$ is equivariantly diffeomorphic to that of $\mathbb{C}_\times$ on $\mathbb{C}^{n+1} \setminus 0$ by

$$x \cdot (z_1, \ldots, z_n, z_{n+1}) = (xz_1, \ldots, xz_n, \overline{x}z_{n+1}).$$

The composition of S with the projection (3.6) maps $(z_1, \ldots, z_n, z_{n+1})$ to the line $[z_1, \ldots, z_n, \overline{z}_{n+1}]$, ensuring that the facial bundles $\rho(D_r)$ are given by $\overline{\zeta}(n)$ for $1 \leq r \leq n$, and $\zeta(n)$ for $r = n+1$. The omniorientation corresponding to (4.2) induces a stably complex structure $\tau(CP^n) \oplus \mathbb{R}^2 \cong n\,\overline{\zeta}(n) \oplus \zeta(n)$.

A second dicharacteristic ℓ arises by setting $\ell(D_{n+1}) = (-1, -1, \ldots, -1)$; the corresponding omniorientation induces $\tau(CP^n) \oplus \mathbb{C}^1 \cong (n+1)\overline{\zeta}(n)$, which stabilises the structure of CP^n considered as an algebraic variety. Both ℓ and ℓ' represent the characteristic map of Example 2.7. When $n = 1$, the structure induced by ℓ represents a generator of Ω^U_2, whereas that induced by ℓ' extends over the 3-disk, and represents zero.

Example 4.3. For B_n as in Example 2.8, we note that $W^{\mathbb{R}}(I^n)$ is $(\mathbb{R}^{\{0,1\}}_{\geqslant} \setminus 0)^n$, and therefore that $W(I^n)$ is isomorphic to $(\mathbb{C}^2 \setminus 0)^n$. The characteristic map of Example 2.8 is represented by the dicharacteristic

$$\ell(C^\epsilon_r) = \begin{cases} (0, \ldots, 0, -1, 0, \ldots, 0) & \text{if } \epsilon = 0 \\ (-1, \ldots, -1, 0, \ldots, 0) & \text{if } \epsilon = 1, \end{cases} \tag{4.4}$$

for all $1 \leq r \leq n$ (where the nonzero elements are in positions r and $1, \ldots, r$ respectively). Thus $K(\ell)$ is the n-dimensional subtorus

$$\{(t_1, t_1^{-1}t_2, \ldots, t_r, t_r^{-1}t_{r+1}, \ldots, t_{n-1}, t_{n-1}^{-1}t_n, t_n, t_n^{-1})\}$$

of T^{2n}. Since the normal space to I^n in $\mathbb{R}^{2n}_{\geqslant}$ is spanned by the n vectors $(0, \ldots, 0, 1, 1, 0, \ldots, 0)$ (nonzero only in positions $2r$ and $2r+1$), the action of $H(\ell)$ on $W(I^n)$ is equivariantly diffeomorphic to that of $(\mathbb{C}_\times)^n$ on $(\mathbb{C}^2 \setminus 0)^n$ by

$$(x_1, \ldots, x_n) \cdot (z_1, w_1, \ldots, z_n, w_n) = (x_1 z_1, \overline{x}_1 x_2 w_1, \ldots, x_n z_n, \overline{x}_n w_n).$$

The composition of S with the projection (3.6) maps $(z_1, w_1, \ldots, z_n, w_n)$ to the bounded flag for which L_r is spanned by the unit vector l_r, given by $\overline{z}_r \overline{w}_r \ldots \overline{w}_n e_r + \lambda l_{r+1}$ (where e_r is the rth basis vector, λ is the normalising factor, and $l_{n+1} = e_{n+1}$), for each $1 \leq r \leq n+1$. Since $H(\ell)$ acts on $\mathbb{C}^{C^0_r}$ by multiplication by x_r, and on $\mathbb{C}^{C^1_r}$ by multiplication by $\overline{x}_r x_{r+1}$, the associated facial bundles $\rho(C^0_r)$ and $\rho(C^1_r)$ are given by $\gamma_r(n)$ and $\rho_r(n)$ respectively. The omniorientation corresponding to (4.4) therefore induces a stably complex structure $\tau(B_n)\oplus\mathbb{R}^{2n}\cong\oplus_{r=1}^n(\gamma_r(n)\oplus\rho_r(n))$. When combined with the canonical trivialisation of $\gamma_1(n)\oplus_{r=1}^n\rho_r(n)$, this reduces to the bounding structure of Example 2.8.

Example 4.5. For $B_{i,j}$ as in Example 2.9, we note that $W(I^i\times\Delta^{j-1})$ is isomorphic to $(\mathbb{C}^2\setminus 0)^i\times\mathbb{C}^j\setminus 0$. The characteristic map of Example 2.9 is represented by the dicharacteristic

$$\ell(E_r^\epsilon)=\begin{cases}(0,\dots,0,-1,0,\dots,0,1,0,\dots,0) & \text{if } \epsilon=0\\ (-1,\dots,-1,-1,0,\dots,0,1,\dots,1,0,\dots,0) & \text{if } \epsilon=1\end{cases}\tag{4.6}$$

for all $1\le r\le n$ (where the nonzero elements are in positions r and $i+r$, and $1,\dots,r$ and $i+1,\dots,i+r-1$ respectively), and

$$\ell(E_s)=\begin{cases}(0,\dots,0,1,0,\dots,0) & \text{for } 1\le s\le j-1\\ (0,\dots,0,-1,\dots,-1) & \text{for } s=j\end{cases}$$

(where the nonzero elements are in positions $i+s$ and $i+1,\dots,i+j-1$ respectively). Thus $K(\ell)$ is the $(i+1)$-dimensional subtorus

$$\{(t_1,t_1^{-1}t_2,\dots,t_{i-1},t_{i-1}^{-1}t_i,t_i,t_i^{-1},t_1^{-1}t_2t,\dots,t_{i-1}^{-1}t_it,t_i^{-1}t,t,\dots,t)\}$$

of T^{2i+j}. Combining Examples 4.1 and 4.3, we deduce that the associated facial bundles $\rho(E_r^0)$ and $\rho(E_r^1)$ are given by $\gamma_r(i)$ and $\rho_r(i)\otimes\overline{\zeta}$ respectively, for each $1\le r\le i$, and that $\rho(E_s)$ is given by $\overline{\zeta}$ for each $1\le s\le j$. The omniorientation corresponding to (4.6) therefore induces a stably complex structure

$$\tau(B_{i,j})\oplus\mathbb{R}^{2(i+1)}\cong\oplus_{r=1}^i\big(\gamma_r(i)\oplus(\rho_r(i)\otimes\overline{\zeta})\big)\oplus j\overline{\zeta}\,.$$

Adding $\gamma_1(i)\otimes\overline{\zeta}$ to both sides and applying the canonical trivialisation of $\gamma_1(i)\oplus_{r=1}^i\rho_r(i)$, we obtain the stably complex structure of Example 2.9 and [5].

We also wish to consider products of these examples, and find it equally convenient to discuss the general case. We assume given omnioriented toric manifolds (M^{2n},α,π,P^n) and (N^{2n},β,μ,Q^n), with corrresponding dicharacteristic maps ℓ_M and ℓ_N. The facets of $P^n\times Q^n$ are of the form $E\times Q^n$ and $P^n\times F$, where E and F range over $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ respectively, so we may define a product dicharacteristic $\ell_{M\times N}\colon\mathcal{F}(P\times Q)\to T^n\times T^n$ by assigning values $\ell_M(E)\times 1$ to $E\times Q^n$ and $1\times\ell_N(F)$ to $P^n\times F$. This corresponds to the product omniorientation of $(M^{2n}\times N^{2n},\alpha\times\beta,\pi\times\mu,P^n\times Q^n)$.

Proposition 4.7. *The stably complex structure induced on $M^{2n}\times N^{2n}$ by the product omniorientation is homotopic to the product of the structures induced by the omniorientations of M^{2n} and N^{2n}.*

Proof. By definition, there is a canonical diffeomorphism between $W(P\times Q)$ and $W(P)\times W(Q)$, and a canonical isomorphism between $H(\ell_{M\times N})$ and $H(\ell_M)\times H(\ell_N)$ which preserves their respective actions on $\mathbb{C}^{\mathcal{F}(P)\sqcup\mathcal{F}(Q)}$ and $\mathbb{C}^{\mathcal{F}(P)}\times\mathbb{C}^{\mathcal{F}(Q)}$. We obtain a diffeomorphism $(M^{2n}\times N^{2n})_\bullet\to M^{2n}_\bullet\times N^{2n}_\bullet$ which repects the induced stably complex structures, and the result therefore follows by pullback along inverse preferred sections, as in Theorem 3.16. $\square$

In [5], we showed that the $B_{i,j}$ are multiplicative generators for the complex cobordism ring Ω_*^U when invested with the stably complex structure of Example 4.5. So every $2n$-dimensional complex cobordism class may be represented by a disjoint union of products

$$B_{i(1),j(1)} \times B_{i(2),j(2)} \times \cdots \times B_{i(t),j(t)}, \tag{4.8}$$

where $\sum_{k=1}^{t}(i(k)+j(k))-2t=n$. Each such component is a toric manifold with the product toric structure. This result is the substance of [5] and may now be enriched by combining Example 4.5 with Proposition 4.7 to confirm that the stably complex structures in question are induced by omniorientations, and are therefore also preserved by the torus action.

To give genuinely toric representatives (which are, by definition, connected) for each cobordism class of dimension >2, it remains only to replace the disjoint union of products (4.8) with their connected sum. This we do in Section 6 below.

5. Cobordism calculations

We now outline certain consequences of the results of Section 3. Our aim is to adapt Davis and Januszkiewicz's programme for computing the integral homology and cohomology of a compact toric manifold (M^{2n},α,π,P^n), so as to apply directly to the complex bordism groups $\Omega_*^U(M^{2n})$ and the complex cobordism ring $\Omega_U^*(M^{2n})$. As always, we work with a fixed omniorientation. We begin by summarising a few prerequisites concerning the bordism and cobodism groups of a CW complex X of finite type. These may be found, for example, in the books of Stong [19] and Switzer [21].

When the cells of X lie only in even dimensions the Atiyah-Hirzebruch spectral sequence collapses, and confirms that generators for the free Ω_*^U-module $\Omega_*^U(X)$ are given by the bordism classes of any set of singular, stably complex manifolds whose top cells correspond to the cells of X. The groups $\Omega_U^*(X)$ are then the $\mathrm{Hom}_{\Omega_*^U}$-duals; when X is itself a stably complex manifold, the multiplicative structure of $\Omega_U^*(X)$ may be extracted from the intersection theory of the generating set by Poincaré duality.

With these considerations in mind, we follow the opening gambit of [8] by constructing a cell decomposition for M^{2n}. This depends on choosing a generic direction in the ambient $\mathbb{R}^n$, and so determining an orientation for each edge of the 1-skeleton of the polytope P^n; the 1-skeleton becomes a directed graph $Di(P)$. Any vertex v has indegree $m(v)$ and outdegree $n-m(v)$ for some integer $0\le m(v)\le n$, and the $m(v)$ inward edges define an $m(v)$-dimensional face G_v of P^n. We write $\widehat{G}_v$ for the subspace obtained by deleting all faces of G_v disjoint from v, which is therefore diffeomorphic to $\mathbb{R}_{\geqslant}^{m(v)}$, and write $e_v\subset M^{2n}$ for the subspace $\pi^{-1}(\widehat{G}_v)$. Since e_v may be identified with $\mathbb{C}^{m(v)}$ it is a $2m(v)$-dimensional cell, and lies within the open set U_v of Section 2; in fact e_v and U_v coincide precisely when v is the sink of $Di(P^n)$, in which case e_v has dimension $2n$. The resulting decomposition of M^{2n} has one cell for each vertex of P^n, and all the cells are even dimensional. The closure of e_v is the facial submanifold $X(G_v)^{2m(v)}$, which inherits the stably complex structure of Theorem 3.16.

Proposition 5.1. *The Ω_*^U-module $\Omega_*^U(M^{2n})$ is generated by the inclusions of the facial submanifolds $X(G)^{2(n-k)}$; none of these is null-cobordant, but they are subject to nontrivial linear relations.*

Proof. The first statement follows from our introductory remarks, in view of the cell decomposition defined above. Since there are m submanifolds $X(F)^{2(n-1)}$, but only $m-n$ two-cells in the decomposition, Poincaré duality shows that there are n linear relations amongst the cobordism classes of the inclusions. $\square$

Proposition 5.1 highlights the remarkable fact that $\Omega_*^U(M^{2n})$ is spanned by *embedded* submanifolds, each of which is equipped with the restricted stably complex structure and is itself a toric manifold.

The omniorientation determines m cobordism Chern classes $c_1(\rho(F))$, each lying in $\Omega_U^2(M^{2n})$ and Poincaré dual to the inclusion $X(F)^{2(n-1)} \subset M^{2n}$ by construction of $\rho(F)$. By transversality, any product $c_1(\rho(F_1))\cdots c_1(\rho(F_k))$ is Poincaré dual to the inclusion of the facial submanifold $X(F_1 \cap \cdots \cap F_k)$; if the intersection of the facets is empty, the bordism and cobordism classes vanish together. So the lattice $\mathfrak{L}_X(M^{2n})$ maps into both $\Omega_*^U(M^{2n})$ and $\Omega_U^*(M^{2n})$.

We deduce that the Ω_*^U-algebra $\Omega_U^*(M^{2n})$ is generated by the Chern classes $c_1(\rho(F))$, and is specified multiplicatively by the ideal of relations amongst them. To compute this ideal, we recall Davis and Januszkiewicz's space BP^n, which depends only on the polytope P^n; all its cells are in even dimensions, and the description of its cohomology extends immediately to complex cobordism. Thus $\Omega_U^*(BP^n)$ is isomorphic to the Stanley-Reisner Ω_*^U-algebra of P^n, which is the quotient of the polynomial algebra $\Omega_*^U[x_F : F \in \mathcal{F}]$ by a certain ideal I, generated by those squarefree monomials $\prod_{\mathcal{E}} x_F = x_{\mathcal{E}}$ for which $\cap_{\mathcal{E}} F$ is empty.

Since BP^n is homotopy equivalent to the Borel construction $ET^n \times_{T^n} M^{2n}$, there is a fibration

$$T^n \to M^{2n} \xrightarrow{j} BP^n, \tag{5.2}$$

classified by a map $l\colon BP^n \to BT^n$. The map j pulls each cobordism class x_F back to the Chern class $c_1(\rho(F))$, whilst l pulls the ith Chern class c_1 back to some element λ_i, both in cohomology and complex cobordism, for $1 \le i \le n$. Considered as a homomorphism on 2-dimensional generators, we may identify $l^*\colon \mathbb{Z}^n \to \mathbb{Z}^{\mathcal{F}}$ with the dual of the dicharacteristic of $(M^{2n}, \alpha, \pi, P^n)$; in this setting we abuse notation by interpreting λ_i as an element of the polynomial algebra $\Omega_U^*[x_F : F \in \mathcal{F}]$.

We then compare the Serre spectral sequence

$$H^*(BP^n; \Omega_U^*(T^n)) \Longrightarrow \Omega_U^*(M^{2n})$$

of (5.2) with the corresponding spectral sequence for the universal principal fibration, which pulls back to the former along l. The only differential is d_2, which annihilates all products of one dimensional elements. We deduce the following result, to be compared with the Danilov-Jurkiewicz Theorem [8] describing the integral cohomology of toric varieties.

Proposition 5.3. *Given any omnioriented toric manifold $(M^{2n}, \alpha, \pi, P^n)$, the cobordism ring $\Omega_U^*(M^{2n})$ is isomorphic to*

$$\Omega_U^*[x_F : F \in \mathcal{F}] \;/\; (I + J),$$

where J denotes the homogeneous ideal generated by the λ_i, and the elements x_F depend on the omniorientation.

In Proposition 5.3, each x_F corresponds to the Chern class $c_1(\rho(F))$. Reversing the facial orientation of a single $X(F)^{2(n-1)}$ therefore applies the inverse of the universal formal group law to x_F. This is linked by Poincaré duality to the effect on bordism theory, which manifests itself in a change of stably complex structure on M^{2n}, and on those submanifolds $X(G)^{2(n-1)}$ for which F meets G. The manifolds themselves remain unaltered.

In the case of bounded flag manifolds, we obtained a result equivalent to Proposition 5.3 in [4]. We did not, however, specify an omniorientation there, but worked instead with the stably complex structure of Example 2.8.

6. Connected sums

In order to construct connected sums of omnioriented compact toric manifolds, we introduce an operation of connected sum for simple polytopes equipped with extra combinatorial data. We work in dimensions ≥ 2, and deal separately with the degenerate case $n = 1$ at the end. Wherever practicable we write $m(P)$ for the number of facets of P^n and $q(P)$ for the number of vertices.

Before we begin we introduce a polyhedral template Γ^n, which is the intersection of n halfspaces in $\mathbb{R}^n$. Strictly speaking, it fails to qualify as a simple n-polyhedron because $n < n+1$, but no contradiction arises from retaining the associated terminology, and we do so for convenience. We embed the standard $(n-1)$-simplex Δ^{n-1} in the subspace $\{x : x_1 = 0\}$ of $\mathbb{R}^{n-1}$, and construct Γ^n by taking cartesian products with the first coordinate axis. Its facets G_r therefore have the form $\mathbb{R} \times D_r$, for $1 \leq r \leq n$. Both Γ^n and G_r are divided into positive and negative halves, determined by the sign of the coordinate x_1.

Given simple polytopes P^n and Q^n in $\mathbb{R}^n$, we assume that respective vertices v and w are distinguished. In addition, we order the facets of P^n meeting in v as E_r, and the facets of Q^n meeting in w as F_r, for $1 \leq r \leq n$. Recalling the notation of Section 3, we write $\mathcal{C}_v$ and $\mathcal{C}_w$ for the complementary sets of facets; those in $\mathcal{C}_v$ avoid v, and those in $\mathcal{C}_w$ avoid w. Their cardinalities are $m(P) - n$ and $m(Q) - n$ respectively.

We now select a projective transformation ϕ_P which maps v to $x_1 = +\infty$, and embeds P^n in Γ^n so as to satisfy two conditions; firstly, that the hyperplane defining E_r is identified with the hyperplane defining G_r, for each $1 \leq r \leq n$, and secondly, that the images of the hyperplanes defining $\mathcal{C}_v$ meet Γ^n in its negative half. This may be achieved, for example, by considering the composition $T \cdot \phi'_P$, where ϕ'_P is an affine equivalence mapping v and its vertex figure to $(1, 0, \ldots, 0)$ and Δ^n respectively, and T is defined by $T(x) = x/(1 - x_1)$. We choose ϕ_Q similarly; it maps w to $x_1 = -\infty$, and identifies the hyperplanes defining F_r and G_r in such a way that the images of the hyperplanes defining $\mathcal{C}_w$ meet Γ^n in its *positive* half. We define the *connected sum* $P^n \#_{v,w} Q^n$ of P^n

at v and Q^n at w to be the simple convex n-polytope determined by all these hyperplanes. It is defined only up to combinatorial equivalence; moreover, different choices for either of v and w, or either of the orderings for E_r and F_r, are likely to affect the combinatorial type. When the choices are clear, or their effect on the result irrelevant, we use the abbreviation $P^n \# Q^n$.

The face lattice $\mathfrak{L}_F(P \# Q)$ is obtained from $\mathfrak{L}_F(P) \cup \mathfrak{L}_F(Q)$ by identifying E_r with F_r for $1 \leq r \leq n$; we write the result as G_r, and partition the facets as

$$\mathcal{F}(P \# Q) = \mathcal{C}_v \cup \{G_r : 1 \leq r \leq n\} \cup \mathcal{C}_w. \tag{6.1}$$

The vertices of $P^n \# Q^n$ are the union of those of P^n and Q^n, omitting v and w. Thus $m(P \# Q) = m(P) + m(Q) - n$, and $q(P \# Q) = q(P) + q(Q) - 2$.

By way of illustration, we consider the connected sum $\Delta^n \#_{v,w} Q^n$, noting that the symmetry of the simplex guarantees that the result is independent of the choice of v. We take v to be 0, so that $\mathcal{C}_v = \{D_{n+1}\}$, and assume that ϕ_{Δ^n} identifies D_r with G_r, for each $1 \leq r \leq n$. So $\phi_{\Delta^n}(\Delta^n)$ consists of Γ^n, truncated in its negative half by a single hyperplane H corresponding to the image of D_{n+1}. Applying ϕ_Q^{-1}, we deduce that the connected sum is combinatorially equivalent to the polytope obtained from Q^n by including an extra hyperplane in the defining set. Such an H must isolate w, but no other vertex, and we interpret its inclusion as *pruning* Q^n at w. We write

$$\Delta^n \#_{v,w} Q^n \equiv \Pi_w(Q^n), \tag{6.2}$$

where Π_w denotes the appropriate pruning operator.

In order to generalise this example to products of simplices, we need a pruning operator Π_F for each face F of Q^n; it is defined in the obvious fashion, and detaches F from Q^n by any hyperplane which separates the vertices of F from the complementary vertices of Q^n. Such operators obey two simple rules, which lead to Theorem 6.5 below. Firstly, for any product $P^m \times Q^n$ and any face E of P^m, there are combinatorial equivalences

$$\begin{aligned} \Pi_E(P^m) \times Q^n \equiv \Pi_{E\times Q}(P^m \times Q^n) \quad \text{and} \\ P^m \times \Pi_F(Q^n) \equiv \Pi_{P\times F}(P^m \times Q^n). \end{aligned} \tag{6.3}$$

Secondly, for any product of simplices $\Delta^m \times \Delta^{n-m}$ with distinguished vertex v there is a face G of Δ^n, a vertex v' not in G, and a combinatorial equivalence

$$\Delta^m \times \Delta^{n-m} \equiv \Pi_G(\Delta^n) \tag{6.4}$$

which maps v to v'.

We now turn to arbitrary products $\Delta^{m_1} \times \cdots \times \Delta^{m_k}$, where $m_1 + \cdots + m_k = n$.

Theorem 6.5. *Given any simple polytope Q^n, there is a combinatorial equivalence*

$$(\Delta^{m_1} \times \cdots \times \Delta^{m_k}) \#_{v,w} Q^n \equiv \Pi_{F_1}(\ldots(\Pi_{F_k}(Q^n))\ldots)$$

for some sequence F_i of products of simplices.

Proof. We first extend (6.4) to k-fold products by induction. For $k \geq 3$, the inductive hypothesis provides a combinatorial equivalence

$$(\Delta^{m_1} \times \cdots \times \Delta^{m_{k-1}}) \times \Delta^{m_k} \equiv \Pi_{G_1}(\ldots(\Pi_{G_{k-2}}(\Delta^{n-m_k}))\ldots) \times \Delta^{m_k},$$

where the G_i are products of simplices; iterating (6.3) replaces the right-hand expression by

$$\varPi_{G_1\times\varDelta^{m_k}}(\ldots(\varPi_{G_{k-2}\times\varDelta^{m_k}}(\varDelta^{n-m_k}\times\varDelta^{m_k}))\ldots),$$

and applying (6.4) yields $\varPi_{F_1}(\ldots(\varPi_{F_{k-1}}(\varDelta^n)\ldots)$, where the F_i are products of simplices as required. As in (6.4), we may ensure that v corresponds to one of the original vertices of $\varDelta^n$ (for which we retain the label v) under the resulting equivalence.

Our connected sum is therefore combinatorially equivalent to

$$\varPi_{F_1}(\ldots(\varPi_{F_{k-1}}(\varDelta^n))\ldots)\,\#_{v,w}\,Q^n. \tag{6.6}$$

Since none of the faces F_i contains the vertex v, they may be identified with the corresponding faces of $\varDelta^n\#_{v,w}Q^n$; in other words, (6.6) is combinatorially equivalent to $\varPi_{F_1}(\ldots(\varPi_{F_{k-1}}(\varDelta^n\#_{v,w}Q^n))\ldots)$, and the result follows by final appeal to (6.2). □

We may now construct the connected sum of omnioriented toric manifolds, assumed henceforth to be compact. Given (M^{2n},α,π,P^n) with fixed point x projecting to the vertex v of P^n, and (N^{2n},β,μ,Q^n), with fixed point y projecting to the vertex w of Q^n, we suppose that the associated dicharacteristics are ℓ_M and ℓ_N respectively. We partition the facets of $P^n\#Q^n$ as in (6.1).

Lemma 6.7. *Up to θ-translation, we may assume that ℓ_M identifies T^{F_r} with the rth coordinate subtorus T_r, for each $1\le r\le n$.*

Proof. Since the subtori $T(F_r)$ generate T^n, we may define an automorphism ψ of T^n by mapping $T(F_r)$ onto T_r, preserving orientation, for each $1\le r\le n$. We conclude by replacing ℓ_M with $\psi\cdot\ell_M$, and appealing to Proposition 3.1. □

Applying Lemma 6.7 to both ℓ_M and ℓ_N allows us to combine them into a dicharacteristic

$$\ell_{\#}(F)=\begin{cases}\ell_M(F) & \text{for } F\in\mathcal{C}_v\\ T_k & \text{for } F=G_r \text{ and } 1\le r\le n\\ \ell_N(F) & \text{for } F\in\mathcal{C}_w\end{cases} \tag{6.8}$$

on $P^n\#_{v,w}Q^n$.

We then define the equivariant connected sum

$$(M^{2n}\#_{x,y}N^{2n},\alpha\#\beta,\pi\#\mu,P^n\#_{v,w}Q^n)$$

to be the omnioriented toric manifold derived from (6.8). Since $P^n\#_{v,w}Q^n$ is determined only up to combinatorial equivalence, we need Corollary 3.11 to ensure that our connected sum is well-defined. Its equivariant diffeomorphism type depends on the choice of fixed points x and y, as well as the orderings of the facets E_r and F_r. Nevertheless, Corollary 6.10 below shows that the properties we require of the induced stably complex structure are suitably invariant.

Theorem 6.9. *The manifold $M^{2n}\#_{x,y}N^{2n}$ is diffeomorphic to the connected sum of M^{2n} and N^{2n}; furthermore, the diffeomorphism identifies the stably complex structure induced by $\ell_{\#}$ with the connected sum of those induced on M^{2n} by ℓ_M and on N^{2n} by ℓ_N.*

Proof. We consider the halfspace H_ϵ of $\mathbb{R}^n$ given by $x_1 \geq \epsilon$ for some small $\epsilon > 0$, and note that its inverse image under the projective transformation ϕ_P is a halfspace H_v, which intersects P^n in a closed neighbourhood N_v of the vertex v. We then apply Lemma 3.10 by choosing $P_1^n = P^n$, $H_1 = H_v$, $P_2^n = P^n \#_{v,w} Q^n$, and $H_2 = H_\#$; we take f to be $\phi_P \colon P_1^n \setminus N(v) \to P^n \#_{v,w} Q^n \cap \{x_1 < \epsilon\}$. We obtain an equivariant diffeomorphism ϕ_P^+ of $M^{2n} \setminus V_x$ into $M^{2n} \#_{x,y} N^{2n}$ (for some invariant neighbourhood V_x of x), which identifies the restrictions of the stably complex structures induced by ℓ_M and $\ell_\#$ respectively. We then repeat the process for Q^n, using the halfspace $x_1 \leq -\epsilon$, and obtain a corresponding diffeomorphism ϕ_Q^+. The images of ϕ_P^+ and ϕ_Q^+ overlap in a collared $(2n-1)$-sphere, and the proof is complete. □

Corollary 6.10. *For any choice of x, y, or orderings of the E_r and F_r, the stably complex structure induced on $M^{2n} \#_{x,y} N^{2n}$ is cobordant to the disjoint union of the structures induced on M^{2n} and N^{2n} respectively.*

Proof. This is a direct consequence of Theorem 6.9, since the connected sum of any two stably complex structures is cobordant to their disjoint union. □

Our main result follows from Corollary 6.10 and Example 4.5.

Theorem 6.11. *In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

It follows from (4.8) and Theorem 6.5 that the base polyhedron of any such manifold may be constructed by pruning Δ^n at an appropriate sequence of products of simplices, up to combinatorial equivalence.

Finally, we address the degenerate case $n = 1$, corresponding to toric manifolds of dimension 2. Any simple polytope of dimension 1 is a 1-simplex, and our definition of connected sum remains valid, yielding $\Delta_1^1 \#_{v,w} \Delta_2^1 = \Delta^1$. We cannot, however, form the connected sum (6.8) of dicharacteristics, because the facets of Δ^1 are its vertices, and the information contained in $\ell_1(v)$ and $\ell_2(w)$ is lost when v and w are deleted. We outline two different approaches to this anomaly, and leave readers to decide on their preference.

We recall that Ω_2^U is isomorphic to $\mathbb{Z}$, and that the cobordism class corresponding to a nonzero integer n may be represented by CP^1 with stably complex structure $\tau \oplus \mathbb{R}^2 \cong \zeta(1)^n \oplus \zeta(1)^n$. According to Example 2.7, this structure is induced by an omniorientation only if $n = \pm 1$. Nevertheless, CP^1 is a toric manifold, and the action of the torus remains compatible with the exotic structure given by other values of n. One option is therefore to assert Theorem 6.11 in this weaker sense when $n = 1$. Alternatively, we might retain the values $\ell_1(v)$ and $\ell_2(w)$ on the connected sum by assigning the information to the 1-simplex (as opposed to its vertices). A second option is therefore to extend the notion of dicharacteristic when $n = 1$, so that Theorem 6.11 holds as stated.

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