A new way to represent links. One-dimensional formalism and untangling technology.

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Abstract

An alternative link representation different from planar diagrams is discussed. Isotopy classes of unordered nonoriented links are realized as central elements of a monoid presented explicitly by a finite number of generators and relations. The group presented by two generators and three relations

$$[[a,b],a^2ba^{-2}]=[[a,b],b^2ab^{-2}]=[[a,b],[a^{-1},b^{-1}]]=1,\quad \text{where } [x,y]=xyx^{-1}y^{-1},$$

is proved to have commutator subgroup isomorphic to the braid group on infinitely many strands.

A new partial algorithm for recognizing is constructed. Experiments show that the algorithm allows to untangle unknots whose planar diagram has hundreds of crossings. Here 'to untangle' means 'to find an isotopy to the circle'.

1 Introduction

Embedding instead of projection.

This paper proposes a new approach to knot theory. While the traditional approach is based on a *projection* of the three-space to a plane, ours uses knots *embedded* in a special subspace $\mathbb{Y}\subset\mathbb{R}^3$ having the form of a three-page book[1]. It is readily seen that any link is equivalent to a link that lies entirely in $\mathbb{Y}$. When projecting links to the plane, one considers only generic projections, which means: no singularities, no triple crossings. Similarly, we use only generic links in $\mathbb{Y}$, which means that the number of the intersection points of the link with the binding line does not decrease under any small perturbation of the link within $\mathbb{Y}$. We also postulate the restriction that any component of the link intersects the binding line. This restriction is essential only for trivial components: if a component of a link in $\mathbb{Y}$ is knotted or linked with the others, then it must intersect the binding line.

One-dimensional formalism instead of a two-dimensional one.

The combinatorial data which is needed to restore a planar link diagram up to isotopy is a four-valent graph with an additional information at nodes (cyclic order of the attached edges and the over/under information). Let us consier the combinatorial structure of a link embedded in the three-page book $\mathbb{Y}$. The intersection of a generic link $L\subset\mathbb{Y}$ with any page of the book $\mathbb{Y}$ consists of a finite number of arcs

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[1]The idea to embed links into a book appeared already in [1]. Recent results on embeddings into a book with unrestricted number of pages can be found in [2] and [3]. I am thankful to S. Chmutov and V. Vershinin for pointing out these papers to me.

resting on the binding line. The arcs do not intersect each other. Therefore, it is enough to know which point is a left endpoint of an arc and which is a right one. This structure is known as well-balanced parentheses string. To restore a link embedded to $\mathbb{Y}$ (up to isotopy), for each intersection point A of L with the binding line, we need to know the following information: in which two pages the arcs attached to A lie and whether A is the left or the right endpoint of those arcs. So, we have 12 possibilities, which means that a generic link in $\mathbb{Y}$ can be encoded by a *word* in a 12-letter alphabet.

It is well known that two planar link diagrams define equivalent links iff one of the diagrams can be obtained from the other by a finite number of Reidemeister moves. In our formalism, links are encoded by words. We find a finite number of local changes in words, which play the same role as the Reidemeister moves do in the formalism of planar diagrams. In this way we obtain a semigroup presented by a finitely many generators and relations.

Unknot recognition problem.

While planar diagrams are convenient for representing knots graphically, our knot representation provides a good way for knot encoding and handling by computer. Consider the problem of recognizing the unknot. An algorithm for doing that was constructed many years ago by Haken [4], but it requires so large a number of operations that it can hardly be run even on a modern computer for a diagram with, say, 10 crossings. Until recently the most effective *practical* solution to the problem was: take a rope knotted according to the given diagram and try to untangle it by hand.

Another algorithm for unknot recognition was constructed recently by J. Birman and M. Hirsch [5]. The algorithm involves representing a link as the closure of a braid. The authors' estimation is that the algorithm can hardly be run for diagrams having 20 crossings. From the practical point of view this is also not satisfactory: having a planar diagram with 20 crossing one can usually decide whether it represents the trivial knot just by looking at it for a few seconds.

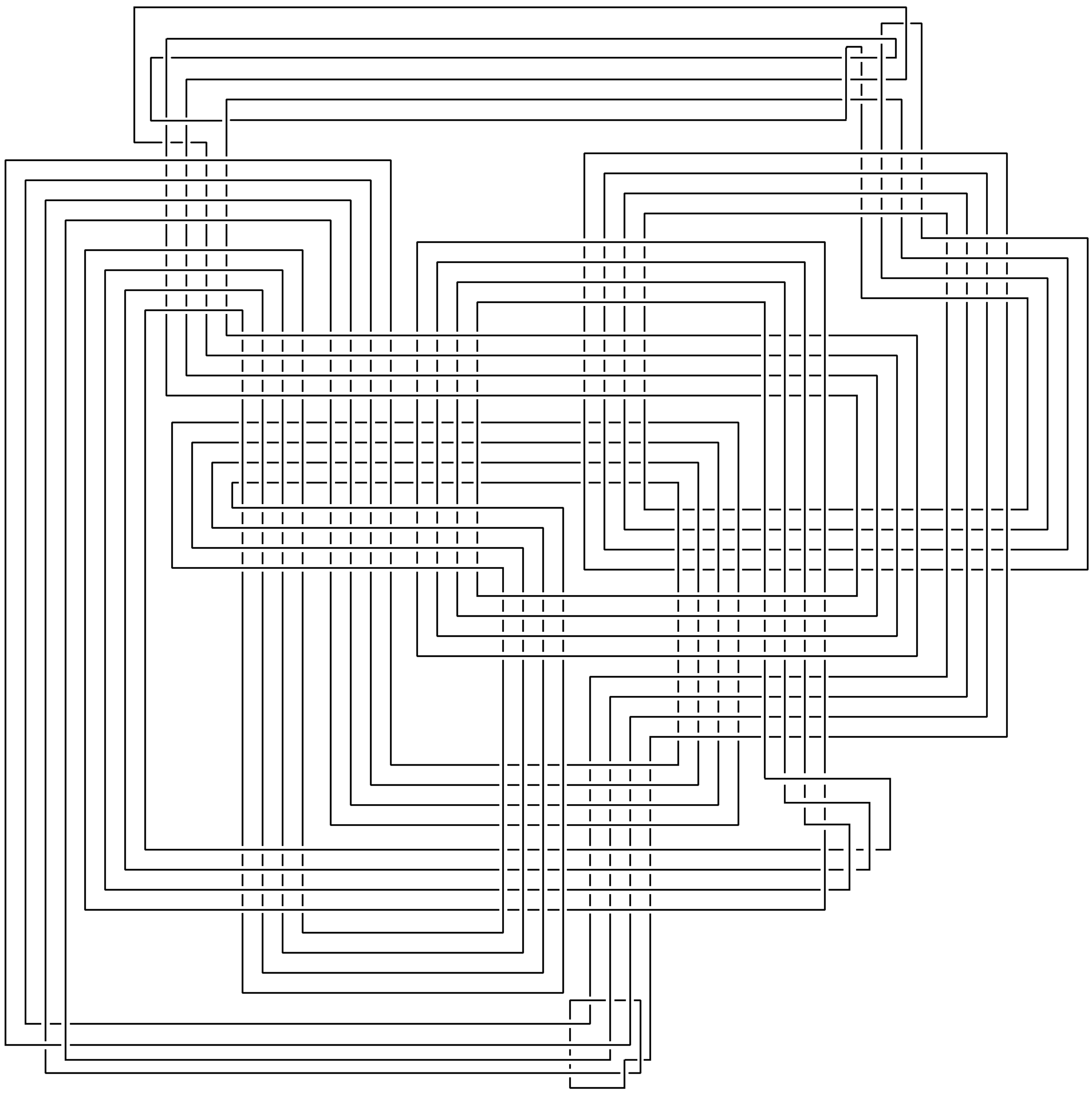
Partial algorithms for unknot recognition.

It seems that a computer cannot compete with a human in unknot recognition if the first implements some *complete* algorithm for unknot recognition. The situation is different for *partial* algorithms. By a partial algorithm we mean a procedure which verifies some sufficient condition for unknottedness. If the condition is not satisfied, such a procedure should stop in finite time. For example, given a planar diagram one can apply all possible Reidemeister moves of type I and II decreasing the number of crossings. This procedure certainly will stop after a finitely many steps and can be implemented easily for very complicated diagrams. If the final diagram has no crossing, then the knot is trivial. If not, nothing can be said about the knot. The monotonic simplification of a planar knot diagram by Reidemeister moves is a very ineffective way for recognizing the unknot. One can easily construct a diagram with about 20 crossings that does not admit a monotonic simplification by Reidemeister moves.

A rather effective partial algorithm for recognition of three-manifolds has been constructed by S. V. Matveev [6]. He uses representation of three-manifolds by their spines and the idea is to simplify the spine by certain transformations fixing the manifold structure. This algorithm was adapted by E. Fominykh for recognizing the unknot. Given a planar diagram, the program first constructs a spine of the knot complement and then simplifies it as much as possible. If, finally, the circle is obtained, then the given knot is reported to be trivial. Due to the fact that the spine of the complement to a knot is a much more complicated object than the planar diagram of the same knot, the algorithm described above takes considerable time to run. The program written by E. Fominykh can be run for a diagram with crossing number 50 and, in experiments, demonstrates ability to recongnize unknots of such complexity.

The partial algorithm for unknot recognition described in this paper is based on the three-page link representation, that is, embeddings of links in the three-page book $\mathbb{Y}$. Having a planar link diagram, we first convert it to a three-page diagram and then simplify it by simple or combined moves (see below). Compared to the Matveev–Fominykh algorithm (which might appear to be more effective) this procedure is very much easier for programming, since it uses very simple combinatorics. It has been tested on a number of unknot diagrams having 20–500 crossings. For all of them a monotonic simplification of the corresponding three-page diagrams has been found. One of the unknots (having 539 crossings) which was successfully untangled by computer is shown in Fig. 1. Other examples, as well as PC executables, can

Figure 1: This unknot has been recognized by computer.



be found at http://mech.math.msu.su/~dynnikov.

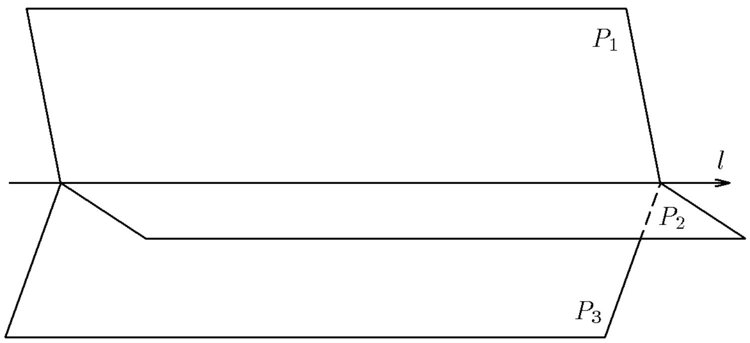
The rest of the paper is arranged as follows. Sections 2–3 contain the basic definitions and preliminary results. The main algebraic results about three-page link representation are given in Sections 4–6. The material of sections 2–6 is published in [7], [8]. Sections 7–9 are mostly devoted to description of the algorithms used in actual computer programs. This description is also published in [9]

I am thankful to S. V. Matveev and S. P. Novikov for their critical comments.

2 Three-page link diagrams.

Let P_1, P_2, P_3 be three half-planes embedded in three-space $\mathbb{R}^3$ so that their boundaries coincide, $\partial P_1 = \partial P_2 = \partial P_3 = l$ (Fig. 2). Denote the union $P_1 \cup P_2 \cup P_3$ by $\mathbb{Y}$. Choose an orientation of the line l.

Figure 2: A three-page book $\mathbb{Y} = P_1 \cup P_2 \cup P_3$.



In this paper by a *polygonal line* we always mean a one-dimensional PL-submanifold of $\mathbb{R}^3$ (not necessarily connected, possibly empty) each of which components is compact.

Definition 1. A *three-page link diagram* is a closed polygonal line $L \subset \mathbb{Y}$ satisfying the following two conditions:

1) *transversality to the straight line* l: the intersections $L \cap l$ is a finite point set, $L \cap l = \{A_1, A_2, \ldots, A_m\}$, and two edges of L attached to A_k lie in different half-planes P_i for any $k \le m$;

2) *monotonicity*: the restriction of the orthogonal projection $\mathbb{R}^3 \to l$ to any connected component of $L \cap P_i$ is a monotone function for any $i = 1, 2, 3$.

An example of a three-page link diagram is shown in Fig. 3.

The number $m = m(L) = \#(L \cap l)$ is called the *complexity* of the diagram L. The connected components of $L \cap P_i$ are called *edges* of L, the points A_i, $i = 1, \ldots m(L)$ are called *vertices* of L. If an edge joins neighboring vertices A_i and A_{i+1} or if it joins A_1 and A_m, then it is said to be *trivial* (we assume that the points in $L \cap l$ are enumerated according to the direction of l).

From condition 2) it follows, in particular, that any connected component of L intersects the line l, which will be referred to as the *binding line*.

Any three-page link diagram defines a link in $\mathbb{R}^3$ in an obvious way, namely, we must take the composition of the embeddings $L \subset \mathbb{Y} \subset \mathbb{R}^3$ and “forget” about $\mathbb{Y}$.

Figure 3: A three-page link diagram.

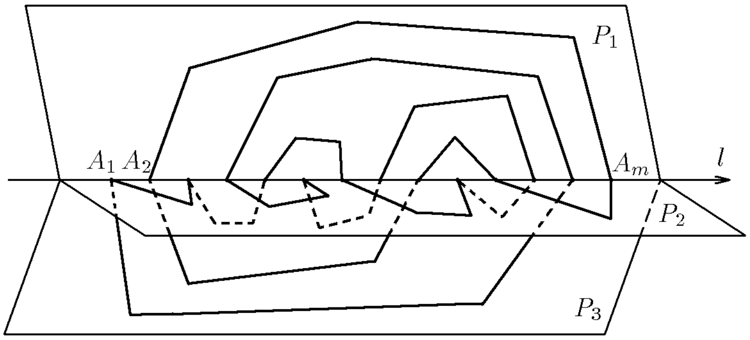
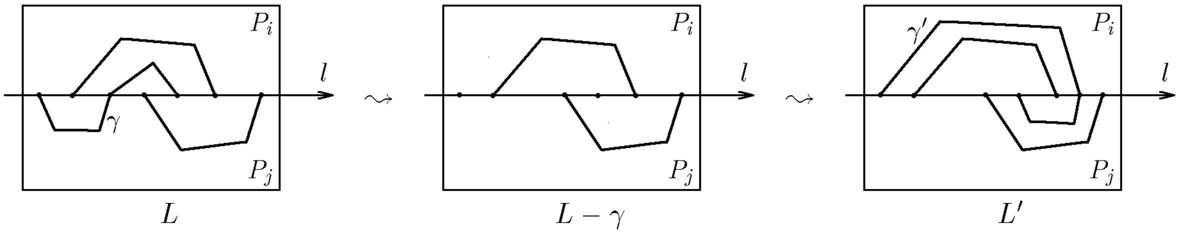


Figure 4: A simple move.



By analogy with the Reidemeister moves, we introduce transformations that lead to isotopic links. Let $1 \leq i, j \leq 3$, $i \neq j$.

Definition 2. We say that a three-page link diagram L' is obtained from L by a *simple move* in the plane $P_i \cup P_j$ if, for some connected component γ of $L \cap (P_i \cup P_j)$ and some connected component γ' of $L' \cap (P_i \cup P_j)$, we have (see Fig. 4)

$$L - \gamma = L' - \gamma'.$$

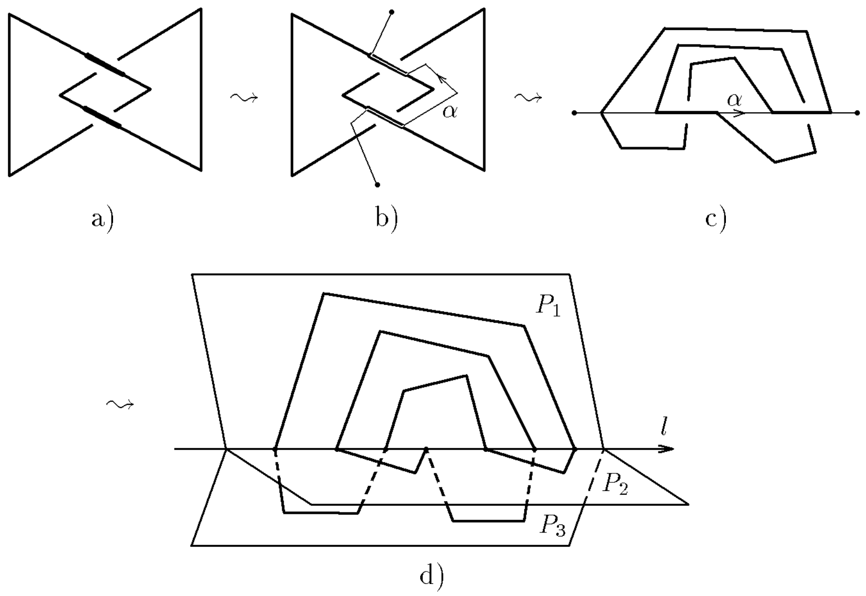
Theorem 1 *Any link can be represented by a three-page link diagram. Two three-page diagrams define isotopic links if and only if they can be obtained from each other by finitely many simple moves.*

The rest of this section is devoted to the proof of the theorem. We can readily see that if L' is obtained from L by a simple move, then the diagrams L and L' represent equivalent links in $\mathbb{R}^3$. Thus, we must only show that simple moves are rich enough to realize any isotopy between two three-page link diagrams. We do this in three steps.

First, we define a special class of three-page diagrams and show how to convert a planar link diagram to a three-page link diagram of this special type representing an isotopic link. The converting procedure is not unique, and the second step is to show that different ways of converting, as well as Reidemeister moves, lead to equivalent three-page diagrams, i.e., diagrams that can be obtained from each other by finitely many simple moves. Finally, we show that any three-page link diagram is equivalent to a special three-page diagram.

Step 1. We say that a three-page link diagram is *special* if each edge in the half-plane P_2 is of length 2, i.e., joins some vertex A_i with the vertex A_{i+2}.

Figure 5: Converting a planar diagram to a three-page diagram.



Let D be a planar link diagram. For each crossing of D let us mark a short segment of the overcrossing arc ('bridge') (Fig. 5, a). Let α be an arbitrary oriented path in the same plane with the following properties:

1) α comes over each bridge exactly once;
2) α intersects the remaining part of the diagram transversally;
3) α has no self-intersections;
4) the endpoints of α are outside D (Fig. 5, b).

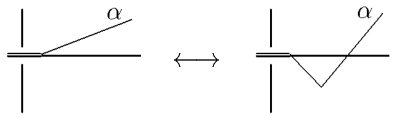
A diagram D endowed with such a curve α will be called a *linearized planar link diagram.*

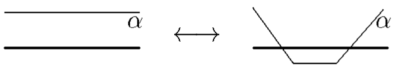
There is an obvious one-to-one correspondence between the isotopy classes of linearized planar diagrams and isotopy classes of special three-page diagrams. To convert a linearized diagram to a three-page diagram one must first deform the plane so that α becomes a straight line segment, then attach a half-plane along the corresponding straight line, and finally push the bridges into the attached half-plane (Fig. 5, c–d).

Step 2. It is easy to prove the following statement.

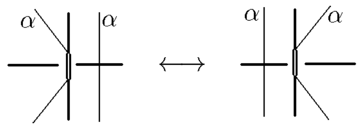
Lemma 1 *Two linearized planar link diagrams (D,α) and (D',α') represent isotopic links if and only if they can be obtained from each other by finitely many isotopies and transformations of the following types:*

type A

A1 

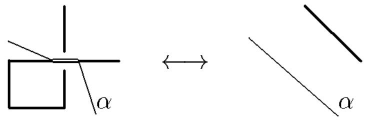
A2 

type B

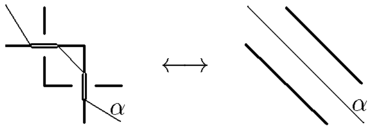


type C (*Reidemeister moves*)

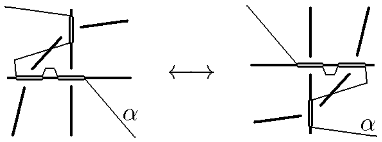
C1



C2



C3



Lemma 2 *The transformations of special three-page diagrams corresponding to the transformations listed in Lemma* 1 *are compositions of simple moves.*

Proof. It can be readily seen that each of the moves of type A and B translates into a single simple move of the corresponding three-page diagrams. Moves C1 and C2 can obviously be represented by a composition of two and three simple moves, respectively.

We demonstrate the sequence of simple moves corresponding to a C3 move, which is the only nontrivial case. We assume that the path α shown in figures in Lemma 1 is directed from north-west to south-east. The first four diagrams are shown in Fig. 6. The fifth, sixth, seventh, and eighths diagrams are symmetric to the fourth, thirth, second, and first diagram, respectively, with respect to a line lying in the plane of P_2 and perpendicular to l. The first arrow is a simple move in the plane $P_2 \cup P_3$, the second in the plane $P_1 \cup P_3$, and the third in the plane $P_1 \cup P_2$. □

Step 3. To complete proof of Theorem 1 it remains to prove the following assertion.

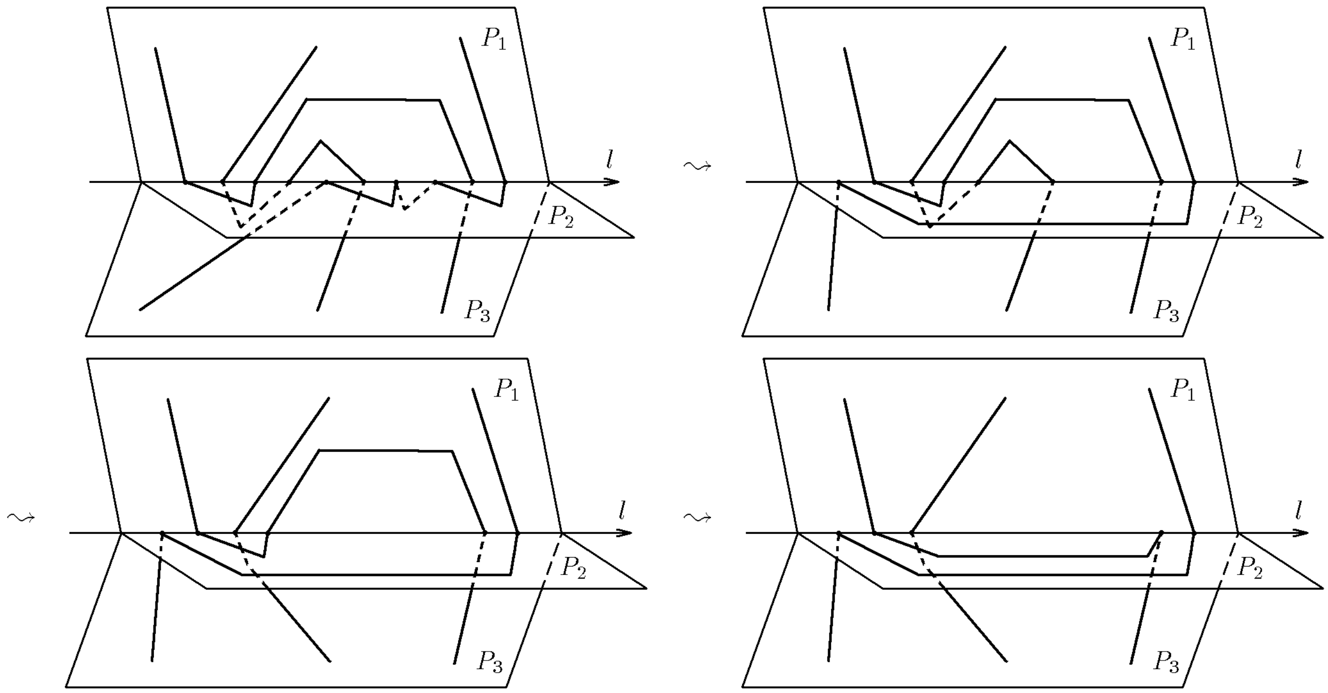
Lemma 3 *Any three-page diagram is equivalent to a special three-page diagram.*

Proof. Obviously, by using simple moves, we can get rid of all trivial edges in the half-plane P_2. Then let us apply the simple moves shown in Fig. 7 until this becomes impossible. It is not hard to see that the process will stop after finitely many steps and the result is a special three-page diagram. □

3 Encoding a three-page diagram.

One may think about a three-page link diagram as a one-dimensional object in the same sense as a usual planar diagram is a two-dimensional one. This can be seen from the following.

Figure 6: Simple moves realizing Reidemeister move of type 3.



Consider the set $\mathbb{A}$ that consists of the following 12 vectors:

$$a_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad b_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad c_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \quad d_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$
$$a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad c_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad d_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$
$$a_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad b_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad c_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad d_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Let a three-page link diagram L be given. As above, denote the vertices in $L \cap l$ by $A_1, \ldots, A_m$, enumerating them in succession according to the direction of the binding line l. To each vertex A_k we assign its *type*, which is a vector $x_k \in \mathbb{A}$ defined by the following rule: for $i \in \{1, 2, 3\}$ the ith coordinate of x_k is

$$x_k^i = \begin{cases} 1, & \text{if } A_k \text{ is the left endpoint of an edge lying in the half-plane } P_i, \\ -1, & \text{if } A_k \text{ is the right endpoint of an edge lying in the half-plane } P_i, \\ 0, & \text{if } A_k \text{ is not an endpoint of an edge lying in } P_i, \end{cases}$$

(see fig. 8). We encode the three-page link diagram L by the following *word in the alphabet* $\mathbb{A}$:

$$w_L = x_1 x_2 \ldots x_m.$$

For example, the three-page diagram of the Whitehead link shown in Fig. 3 is represented by the word

$$a_1\, a_2\, d_1\, a_3\, d_2\, d_1\, d_3\, d_2\, d_2\, d_1\, d_3\, c_2\, c_2\, c_3.$$

Figure 7: Transforming a three-page diagram into a special one.

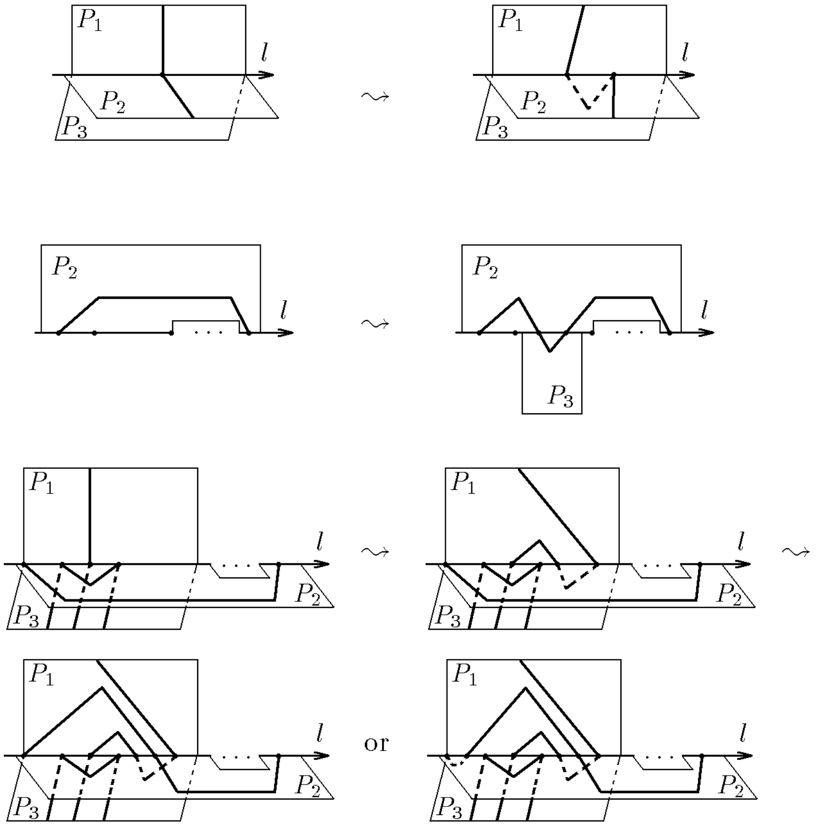
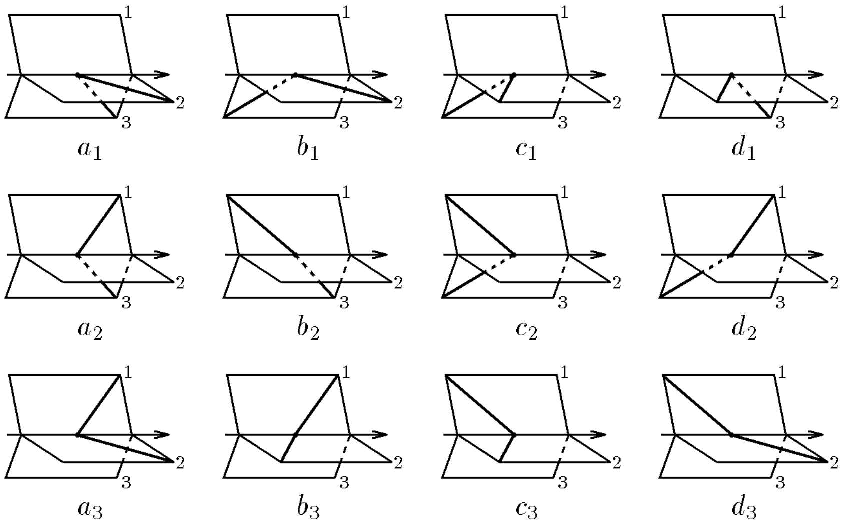


Figure 8: Types of vertices



Denote by W the set of all words in the alphabet $\mathbb{A}$ including the empty word $\varnothing$. We need the following definition.

Definition 3. For $i \in \{1,2,3\}$ a word $w = x_1 \ldots x_m \in W$ is said to be *i-balanced* if $\sum_{k=1}^{m} x_k^i = 0$ and

$$\sum_{k=1}^{p} x_k^i \geq 0 \quad \text{for any } 1 \leq p \leq k.$$

If a word is i-balanced for all $i \in \{1,2,3\}$, then we say that it is *completely balanced.*

In another way, an i-balanced word can characterized in the following way: if in the sequence $x_1^i \ldots x_m^i$ all symbols "1" are replaced by a left parenthesis and all symbols "-1" by a right one, then the obtained parenthesis string is balanced. The next assertion is obvious.

Proposition 1 *A word $w \in W$ if of the form $w = w_L$, where L is a three-page link diagram, if and only if w is completely balanced. Two nonoriented three-page link diagrams L and L' are ambient isotopic in $\mathbb{Y}$ if and only if $w_L = w_{L'}$.*

Thus, the set $W_0 \subset W$ of completely balanced words in the alphabet $\mathbb{A}$ classifies three-page link diagrams up to an ambient isotopy in $\mathbb{Y}$. In the next section we indicate a finite set of local transformations of completely balanced words that generates the equivalence relation for links.

Note that encoding three-page link diagrams by words in $\mathbb{A}$ provides a convenient way to encode ordinary planar diagrams. Indeed, the Gauss code of a link diagram with n crossings occupies an amount of memory of size $O(n \log n)$, whereas the same diagram can be encoded by a word in the *finite* alphabet $\mathbb{A}$ of length $\leq (3n+1)$ using the following statement.

Lemma 4 *There is an injective map* {connected planar diagrams} $\to$ {three-page diagrams} *which sends any planar link diagram of complexity n to a three-page diagram of complexity $\leq 3n+1$.*

The proof of the lemma consists in the direct construction of the map. One such map comes from the proof of Theorem 1. One must find a curve α transforming the given planar diagram into a linearized one and intersecting the diagram transversally no more than once.

In computer programs, we use another method of converting a planar diagram to a three-page one. It gives the same estimation $3n+1$ for the complexity of the latter and is extremely easy for programming. We describe it in Section 7.

4 Local moves.

Having written (isotopy classes of) three-page link diagrams as words in the 12-letter alphabet, it is very desirable to find a finite number of *local* moves such that any simple move can be represented as a composition of local ones. Surprisingly, this is possible! We shall show this in the present section.

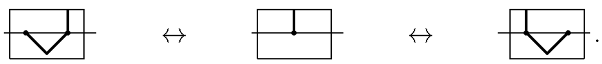
First, we make clear the notion of 'local move'. By a local move we mean a pair of words (u, v) in the alphabet $\mathbb{A}$. Given such a pair, we say that a three-page link diagram L' is obtained from L by the local move (u, v) if the word representing L' is obtained from that representing L by replacing a subword u with the subword v.

The simple moves introduced in Section 2 are not local. The number of simple moves that can be applied to a complicated three-page link diagram is huge. It growths as C^n, where n is the complexity of the diagram. This circumstance increases our chances to simplify an excessively tangled diagram of a not very complicated link by one move, but in order to reveal the algebraic nature of the three-page representation and construct link invariants, we need to reduce the number of allowed moves.

To begin with, we show that any simple move is a composition of *elementary moves* of three types.

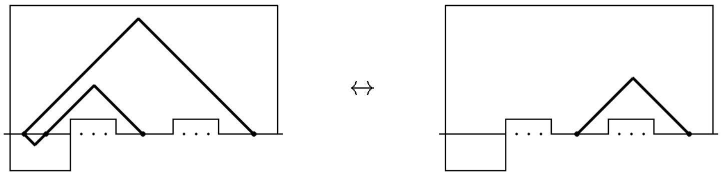
Proposition 2 *Any simple move can be decomposed into a sequence of simple moves of the following three types.*

Type I (inflating/contracting a trivial edge):

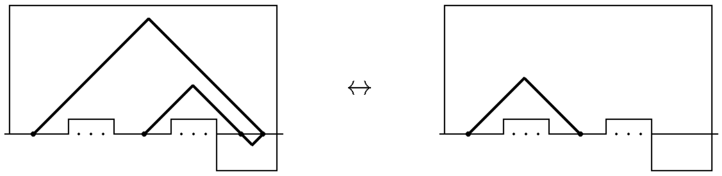


Type II (inserting/removing a trivial edge):

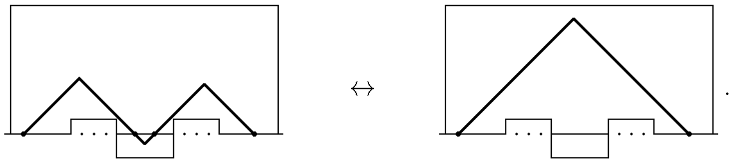
II′



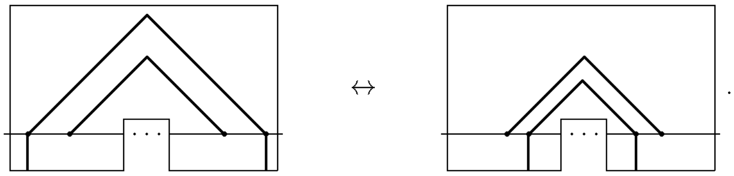
II″

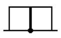
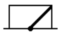
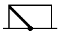


II‴



Type III (interchanging unlinked parallel edges):



In this figures, it is assumed that a tail perpendicular to the binding line  can be directed both to the right and to the left,  and . If only one tail attached to a vertex is indicated, then the other is assumed to be in the half-plane(s) not shown in the figure. For example, each of the type I and type II moves can be interpreted in 4 ways and the type III move can be interpreted in 16 ways.

Proof. One can show that all the simple moves that were used in the proof of Theorem 1 can be decomposed into elementary moves mentioned in Proposition 2. We give an alternative, direct, proof of the claim.

To be definite, assume that we intend to perform a simple move in the plane $\Pi = P_1 \cup P_2$, i.e., to replace a connected component γ of $L \cap \Pi$ by another path $\gamma' \subset \Pi$, $\partial\gamma = \partial\gamma'$. If γ and γ' are isotopic in $\Pi - (L - \gamma)$ (for example, if γ is the only component of $L \cap \Pi$), then there is no problem, namely, finitely many moves of type I and type II must be applied.

The obstacles to an isotopy (with the endpoint fixed) between γ and γ' may be only the connected components $\gamma_1, \ldots \gamma_k$ of $L \cap \Pi$ distinct from γ. In order to transform γ to γ', we must perform isotopies in $\Pi - (\bigcup_i \gamma_i)$ and throw γ over γ_i finitely many times.

A type III move allows one to through γ over γ_i if the latter consists of exactly one edge. Thus, to complete the proof, it remains to reduce the general case to the case in which each component γ_i consits of a edge. To this end, we blow up any vertex $A_k \notin \gamma$ lying at the joining point of two edges one of which belongs to P_1 and the other to P_2 into a trivial edge in the half-plane P_3. □

In what follows we shall translate geometric pictures into algebraic language, using the alphabet $\mathbb{A}$. We now introduce a set of local moves that suffice to produce the equivalence for three-page link diagrams. All the indexes of a_i, b_i, c_i, d_i, P_i are regarded modulo 3.

Theorem 2 *Two completely balanced words from W define equivalent nonoriented unordered links in $\mathbb{R}^3$ if and only if one of the words can be obtained from the other by a finite sequence of the following local changes of subwords:*

$$\text{Type I:} \quad \left\{ \begin{array}{ccccc} a_{i+1}d_{i-1} & \leftrightarrow & a_i & \leftrightarrow & a_{i-1}b_{i+1} \\ d_{i+1}d_{i-1} & \leftrightarrow & b_i & \leftrightarrow & a_{i-1}c_{i+1} \\ d_{i+1}c_{i-1} & \leftrightarrow & c_i & \leftrightarrow & b_{i-1}c_{i+1} \\ a_{i+1}c_{i-1} & \leftrightarrow & d_i & \leftrightarrow & b_{i-1}b_{i+1} \end{array} \right. , \quad i \in \mathbb{Z}_3 \tag{1}$$

$$\text{Type II:} \quad \left\{ \begin{array}{ccc} a_{i-1}b_{i-1}\,w & \leftrightarrow & w\,a_{i-1}b_{i-1} \\ d_{i-1}c_{i-1}\,w & \leftrightarrow & w\,d_{i-1}c_{i-1} \\ d_{i-1}b_{i-1}\,w & \leftrightarrow & w\,d_{i-1}b_{i-1} \end{array} \right. , \quad w \in X_i, \quad i \in \mathbb{Z}_3 \tag{2}$$

$$\text{Type III:} \quad (b_{i+1}d_{i-1}d_{i+1}b_{i-1})\,w \;\leftrightarrow\; w\,(b_{i+1}d_{i-1}d_{i+1}b_{i-1}), \quad w \in X_i, \quad i \in \mathbb{Z}_3, \tag{3}$$

where $X_i = \{a_i,\ b_i,\ c_i,\ d_i,\ b_{i-1}a_id_{i-1},\ b_{i-1}b_id_{i-1},\ b_{i-1}c_id_{i-1},\ b_{i-1}d_id_{i-1}\}$. All the listed transformations preserve the subset of completely balanced words.

For two words $u, v \in W$, we write $u \sim v$ if u can be obtained from v by a finite sequence of local changes of the form (1–3).

Remark 1. Note that here we do not use the moves $b_id_i \leftrightarrow \varnothing \leftrightarrow d_ib_i$, though they might seem natural. Indeed, when inserting the subword b_id_i or d_ib_i into a balanced word, we may obtain a nonbalanced one. However, deleting the subwords b_id_i and d_ib_i is always allowed and we shall show that this deletion can be decomposed into a sequence of moves listed in the theorem.

The remaining part of this section is devoted to the proof of Theorem 2. To begin with, we perform some preliminary calculations.

Lemma 5 *For $i \in \mathbb{Z}_3$ we have*

$$\begin{array}{ccccc} a_id_ib_i & \sim & a_i & \sim & a_ib_id_i \\ b_id_ib_i & \sim & b_i & \sim & a_ib_ic_i \\ b_id_ic_i & \sim & c_i & \sim & d_ib_ic_i \\ a_id_ic_i & \sim & d_i & \sim & d_ib_id_i \end{array} \tag{4}$$

$$\begin{array}{ccc} a_ib_i & \sim & a_{i-1}d_{i-1} \\ d_ic_i & \sim & b_{i-1}c_{i-1} \\ d_ib_i & \sim & b_{i-1}d_{i-1}. \end{array} \tag{5}$$

Proof. Indeed,

$$a_id_ib_i \sim a_ib_{i-1}b_{i+1}b_i \sim a_{i+1}d_{i-1} \sim a_i \sim a_{i-1}b_{i+1} \sim a_id_{i+1}d_{i-1}d_i \sim a_ib_id_i$$

$$b_id_ib_i \sim b_ib_{i-1}b_{i+1}b_i \sim d_{i+1}d_{i-1} \sim b_i \sim a_{i-1}c_{i+1} \sim a_id_{i+1}d_{i-1}c_i \sim a_ib_ic_i$$

$$b_id_ic_i \sim b_ib_{i-1}b_{i+1}c_i \sim d_{i+1}c_{i-1} \sim c_i \sim b_{i-1}c_{i+1} \sim d_id_{i+1}d_{i-1}c_i \sim d_ib_ic_i$$

$$a_i d_i c_i \sim a_i b_{i-1} b_{i+1} c_i \sim a_{i+1} c_{i-1} \sim d_i \sim b_{i-1} b_{i+1} \sim d_i d_{i+1} d_{i-1} d_i \sim d_i b_i d_i$$
$$a_i b_i \sim a_{i-1} b_{i+1} b_i \sim a_{i-1} d_{i-1}$$
$$d_i c_i \sim b_{i-1} b_{i+1} c_i \sim b_{i-1} c_{i-1}$$
$$d_i b_i \sim b_{i-1} b_{i+1} b_i \sim b_{i-1} d_{i-1}.$$

□

Lemma 6 *For any $i \in \mathbb{Z}_3$ and $x \in \{a_i, b_i, c_i, d_i\}$ we have*

$$d_{i-1} b_{i+1} b_{i-1} d_{i+1}\ x \sim x\ d_{i-1} b_{i+1} b_{i-1} d_{i+1}. \tag{6}$$

Proof. Indeed,

$$\begin{aligned}
d_{i-1}b_{i+1}b_{i-1}d_{i+1}x &\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}b_{i+1}d_{i+1}x \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}xb_{i+1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i-1}b_{i-1}d_{i+1}xb_{i+1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i-1}b_{i-1}\ b_i\ b_{i-1}xd_{i-1}\ d_id_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}b_ib_{i-1}\ x\ d_{i-1}d_id_{i-1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i-1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i+1}b_{i+1}d_{i-1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i-1}d_{i+1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}b_{i+1}b_id_{i+1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i-1}d_ib_id_{i+1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i-1}d_{i+1}d_ib_ib_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ x\ b_{i+1}d_{i-1}d_{i+1}b_{i-1}\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i+1}\ b_{i+1}d_{i-1}d_{i+1}b_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}d_{i+1}b_{i+1}b_{i-1}d_{i-1}d_{i+1}b_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}b_{i-1}d_{i-1}d_{i+1}b_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}d_ib_id_{i+1}b_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}d_{i+1}d_ib_ib_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}d_{i+1}d_id_{i+1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i+1}d_{i+1}b_{i-1}\ x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}d_{i-1}b_{i-1}\ b_{i-1}xd_{i-1}\ b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}\ b_{i-1}xd_{i-1}\ d_{i-1}b_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i-1}xd_{i-1}d_{i-1}d_ib_{i-1}d_{i+1} \\
&\sim d_{i-1}b_{i-1}xd_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim xd_{i-1}b_{i-1}d_{i-1}b_{i+1}b_{i-1}d_{i+1} \\
&\sim x\ d_{i-1}b_{i+1}b_{i-1}d_{i+1}
\end{aligned}$$

□

Lemma 7 *For $i \in \mathbb{Z}_3$ we have*

$$\begin{aligned}
b_{i-1}^2 a_i d_{i-1}^2 &\sim b_{i-1}a_id_{i-1}\ d_i^2\ b_{i-1}b_id_{i-1}\ b_i \\
b_{i-1}^2 b_i d_{i-1}^2 &\sim b_{i-1}b_id_{i-1}\ d_i^2\ b_{i-1}b_id_{i-1}\ b_i \\
b_{i-1}^2 c_i d_{i-1}^2 &\sim d_i\ b_{i-1}d_id_{i-1}\ b_i^2\ b_{i-1}c_id_{i-1} \\
b_{i-1}^2 d_i d_{i-1}^2 &\sim d_i\ b_{i-1}d_id_{i-1}\ b_i^2\ b_{i-1}d_id_{i-1}
\end{aligned} \tag{7}$$

Proof. Indeed,

$$\begin{aligned}
b_{i-1}^2a_id_{i-1}^2 &\sim b_{i-1}^2a_id_ib_id_{i-1}^2 \\
&\sim b_{i-1}^2a_{i+1}b_{i+1}b_id_{i-1}^2 \\
&\sim b_{i-1}a_{i+1}b_{i+1}b_{i-1}b_id_{i-1}^2 \\
&\sim b_{i-1}a_id_ib_{i-1}b_id_{i-1}^2 \\
&\sim b_{i-1}a_i\ d_ib_{i-1}b_id_{i-1}\ b_{i+1}\ b_i \\
&\sim b_{i-1}a_ib_{i+1}d_ib_{i-1}b_id_{i-1}b_i \\
&\sim b_{i-1}a_id_{i-1}d_i^2b_{i-1}b_id_{i-1}b_i.
\end{aligned}$$

$$
\begin{array}{rcl}
b_{i-1}^2 b_i d_{i-1}^2 & \sim & b_{i-1}^2 b_i d_i b_i d_{i-1}^2 \\
& \sim & b_{i-1}^2 d_{i+1} b_{i+1} b_i d_{i-1}^2 \\
& \sim & b_{i-1} d_{i+1} b_{i+1} b_{i-1} b_i d_{i-1}^2 \\
& \sim & b_{i-1} b_i d_i b_{i-1} b_i d_{i-1}^2 \\
& \sim & b_{i-1} b_i \ d_i b_{i-1} b_i d_{i-1} \ b_{i+1} \ b_i \\
& \sim & b_{i-1} b_i b_{i+1} d_i b_{i-1} b_i d_{i-1} b_i \\
& \sim & b_{i-1} b_i d_{i-1} d_i^2 b_{i-1} b_i d_{i-1} b_i .
\end{array}
$$

$$
\begin{array}{rcl}
b_{i-1}^2 c_i d_{i-1}^2 & \sim & b_{i-1}^2 d_i b_i c_i d_{i-1}^2 \\
& \sim & b_{i-1}^2 d_i d_{i+1} c_{i+1} d_{i-1}^2 \\
& \sim & b_{i-1}^2 d_i d_{i-1} d_{i+1} c_{i+1} d_{i-1} \\
& \sim & b_{i-1}^2 d_i d_{i-1} b_i c_i d_{i-1} \\
& \sim & d_i d_{i+1} \ b_{i-1} d_i d_{i-1} b_i \ c_i d_{i-1} \\
& \sim & d_i b_{i-1} d_i d_{i-1} b_i d_{i+1} c_i d_{i-1} \\
& \sim & d_i b_{i-1} d_i d_{i-1} b_i^2 b_{i-1} c_i d_{i-1} .
\end{array}
$$

$$
\begin{array}{rcl}
b_{i-1}^2 d_i d_{i-1}^2 & \sim & b_{i-1}^2 d_i b_i d_i d_{i-1}^2 \\
& \sim & b_{i-1}^2 d_i d_{i+1} b_{i+1} d_{i-1}^2 \\
& \sim & b_{i-1}^2 d_i d_{i-1} d_{i+1} b_{i+1} d_{i-1} \\
& \sim & b_{i-1}^2 d_i d_{i-1} b_i d_i d_{i-1} \\
& \sim & d_i d_{i+1} \ b_{i-1} d_i d_{i-1} b_i \ d_i d_{i-1} \\
& \sim & d_i b_{i-1} d_i d_{i-1} b_i d_{i+1} d_i d_{i-1} \\
& \sim & d_i b_{i-1} d_i d_{i-1} b_i^2 b_{i-1} d_i d_{i-1} .
\end{array}
$$

□

The following assertion generalizing Lemma 3 plays the key role in the proof of Theorem 2.

Lemma 8 *If $w \in W$ is an i-balanced word, then there exist a sequence $w_1, w_2, \ldots, w_k$ of elements of X_i such that $w \sim w_1 w_2 \ldots w_k$.*

The converse assertion is obvious. Indeed, if $w_j \in X_i$, $j = 1, \ldots, k$, then the word $w_1 \ldots w_k$ is i-balanced. We shall denote the set of all i-balanced word by $\overline{X}_i$.

PROOF. Let $w \in W$ be i-balanced word. Let us make all the possible changes in w belonging to the following list:

$$
\begin{array}{rcl}
a_{i-1} & \rightsquigarrow & a_i d_{i+1} \\
a_{i+1} & \rightsquigarrow & a_i b_{i-1} \\
c_{i-1} & \rightsquigarrow & b_{i+1} c_i \\
c_{i+1} & \rightsquigarrow & d_{i-1} c_i \\
b_{i+1} & \rightsquigarrow & d_{i-1} d_i \\
d_{i+1} & \rightsquigarrow & b_i b_{i-1}
\end{array}
$$

We obtain a new word w' containing only the letters $a_i, b_i, c_i, d_i, b_{i-1}, d_{i-1}$. Moreover, since the word w was i-balanced, it is not hard to see that the word w' has the following property: if we replace each b_{i-1} by a left parenthesis '(' and each d_{i-1} by a right parenthesis ')', then we obtain an expression with well-balanced parentheses.

Now, to transform the word w' to a composition of words from X_i, we apply the following local changes as long as possible:

$$b_{i-1} d_{i-1} \rightsquigarrow d_i b_i$$

$$b_{i-1} uv \rightsquigarrow b_{i-1} d_{i-1} b_{i-1} uv \rightsquigarrow b_{i-1} u d_{i-1} \ b_{i-1} v, \text{ where } u, v \in X_i$$

$$
\begin{array}{rcl}
b_{i-1}^2 a_i d_{i-1}^2 & \rightsquigarrow & b_{i-1} a_i d_{i-1} \ d_i^2 \ b_{i-1} b_i d_{i-1} \ b_i \\
b_{i-1}^2 b_i d_{i-1}^2 & \rightsquigarrow & b_{i-1} b_i d_{i-1} \ d_i^2 \ b_{i-1} b_i d_{i-1} \ b_i \\
b_{i-1}^2 c_i d_{i-1}^2 & \rightsquigarrow & d_i \ b_{i-1} d_i d_{i-1} \ b_i^2 \ b_{i-1} c_i d_{i-1} \\
b_{i-1}^2 d_i d_{i-1}^2 & \rightsquigarrow & d_i \ b_{i-1} d_i d_{i-1} \ b_i^2 \ b_{i-1} d_i d_{i-1}
\end{array}
$$

As was shown above, all these transformations are compositions of local changes (1–3). Replacing all elements of $\{a_i, b_i, c_i, d_i\}$ by '$\bullet$', b_{i-1} by '(' and d_{i-1} by ')' we obtain the following graphical interpretation of the above transformations:

$$
\begin{array}{rcl}
() & \rightsquigarrow & \bullet\bullet \\
(\bullet\bullet & \rightsquigarrow & (\bullet)(\bullet \\
(\bullet(\bullet) & \rightsquigarrow & (\bullet)((\bullet) \\
((\bullet)\bullet & \rightsquigarrow & ((\bullet))(\bullet \\
((\bullet)(\bullet) & \rightsquigarrow & ((\bullet))((\bullet) \\
((\bullet)) & \rightsquigarrow & (\bullet)\bullet\bullet(\bullet)\bullet \text{ or } \bullet(\bullet)\bullet\bullet(\bullet).
\end{array}
$$

It is clear that the process will stop after finitely many steps. □

We are now ready to check that the local moves (1–3) are sufficient to perform any elementary move listed in Proposition 2. To this end, we write the elementary moves algebraically using the alphabet $\mathbb{A}$.

The elementary moves of type I are local and they are translated exactly into relations (1). The elementary moves of type II are written as follows:

$$a_{i-1}b_{i-1}\, w\, x \sim w\, y \sim a_{i+1}d_{i+1}\, w\, x, \tag{8}$$

where $w \in \overline{X}_i$, $x, y \in \mathbb{A}$, $x^i = -1 = -y^i$, $x^{i+1} = y^{i+1}$, $x^{i-1} = y^{i-1}$;

$$x\, w\, d_{i-1}c_{i-1} \sim y\, w \sim x\, w\, b_{i+1}c_{i+1}, \tag{9}$$

where $w \in \overline{X}_i$, $x, y \in \mathbb{A}$, $x^i = 1 = -y^i$, $x^{i+1} = y^{i+1}$, $x^{i-1} = y^{i-1}$;

$$x\, w\, d_{i-1}b_{i-1} \sim x\, w \sim x\, w\, b_{i+1}d_{i+1}, \tag{10}$$

where $w \in \overline{X}_i$, $x \in \mathbb{A}$, $x^i = 1$.

Let us prove (8). Let $w \sim w_1 \dots w_k$ be the decomposition from Lemma 8. We have

$$
\begin{array}{rcl}
a_{i-1}b_{i-1}\, w\, x & \sim & a_{i-1}b_{i-1}\, w_1 w_2 \dots w_k\, x \\
& \sim & w_1\, a_{i-1}b_{i-1}\, w_2 \dots w_k\, x \\
& \sim & w_1 w_2\, a_{i-1}b_{i-1}\, \dots w_k\, x \\
& & \vdots \\
& \sim & w_1 w_2 \dots w_k\, a_{i-1}b_{i-1}\, x \\
& \sim & w\, a_{i-1}b_{i-1}\, x \\
& \sim & w\, y \\
a_{i-1}b_{i-1}\, w\, x & \sim & a_{i+1}d_{i+1}\, w\, x,
\end{array}
$$

where we used (2), (4), and (5). Relations (9) and (10) can be proved in a similar way. □

To realize the elementary moves of type III as compositions of (1–3) we use the following lemma.

Lemma 9 *We have*

$$
\begin{array}{rcl}
a_{i-1}a_{i+1}\; b_{i+1}d_{i-1}d_{i+1}b_{i-1} & \sim & a_{i+1}a_{i-1} \\
b_{i-1}a_{i+1}\; b_{i+1}d_{i-1}d_{i+1}b_{i-1} & \sim & a_{i+1}b_{i-1} \\
a_{i-1}d_{i+1}\; b_{i+1}d_{i-1}d_{i+1}b_{i-1} & \sim & d_{i+1}a_{i-1} \\
b_{i-1}d_{i+1}\; b_{i+1}d_{i-1}d_{i+1}b_{i-1} & \sim & d_{i+1}b_{i-1}
\end{array} \tag{11}
$$

$$
\begin{array}{rcl}
b_{i+1}d_{i-1}d_{i+1}b_{i-1}\; c_{i-1}c_{i+1} & \sim & c_{i+1}c_{i-1} \\
b_{i+1}d_{i-1}d_{i+1}b_{i-1}\; d_{i-1}c_{i+1} & \sim & c_{i+1}d_{i-1} \\
b_{i+1}d_{i-1}d_{i+1}b_{i-1}\; c_{i-1}b_{i+1} & \sim & b_{i+1}c_{i-1} \\
b_{i+1}d_{i-1}d_{i+1}b_{i-1}\; d_{i-1}b_{i+1} & \sim & b_{i+1}d_{i-1}
\end{array} \tag{12}
$$

Proof. Let $x_1 \in \{a_{i-1}, b_{i-1}\}$, $x_2 \in \{a_{i+1}, d_{i+1}\}$. It follows from (2) and (4–5) that

$$x_1 x_2 b_{i+1} d_{i-1} d_{i+1} b_{i-1} \sim x_1 d_{i-1} x_2 b_{i+1} d_{i+1} b_{i-1} \sim x_1 d_{i-1} x_2 b_{i-1} \sim x_2 x_1 d_{i-1} b_{i-1} \sim x_2 x_1.$$

Relations (12) can be proved in a similar way. □

The elementary moves of type III have the form

$$x_1 x_2\, w\, y_2 y_1 \sim x_2 x_1\, w\, y_1 y_2, \tag{13}$$

where $w \in \overline{X}_i$, $x_1 \in \{a_{i-1}, b_{i-1}\}$, $x_2 \in \{a_{i+1}, d_{i+1}\}$, $y_1 \in \{c_{i-1}, d_{i-1}\}$, $y_2 \in \{c_{i+1}, b_{i+1}\}$.
To prove (13), we again use a decomposition $w \sim w_1 \ldots w_k$, $w_1, \ldots, w_k \in X_i$. Applying Lemma 9, we obtain

$$\begin{array}{rcl}
x_1 x_2\, w\, y_2 y_1 & \sim & x_1 x_2\, w\, b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, y_1 y_2 \\
& \sim & x_1 x_2\, w_1 \ldots w_{k-1} w_k\, b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, y_1 y_2 \\
& \sim & x_1 x_2\, w_1 \ldots w_{k-1}\, b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, w_k\, y_1 y_2 \\
& \sim & x_1 x_2\, w_1 \ldots b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, w_{k-1} w_k\, y_1 y_2 \\
& & \vdots \\
& \sim & x_1 x_2\, b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, w_1 \ldots w_{k-1} w_k\, y_1 y_2 \\
& \sim & x_1 x_2\, b_{i+1} d_{i-1} d_{i+1} b_{i-1}\, w\, y_1 y_2 \\
& \sim & x_2 x_1\, w\, y_1 y_2.
\end{array}$$

□

5 Three-page tangles and three-page braids

In this section we formulate the main algebraic result of this paper.

In the previous section we considered only completely balanced words in the alphabet $\mathbb{A}$. Factorizing it by transformations (1–3), we obtained a semigroup that has a poor structure: it is commutative, since the concatenation of completely balanced words corresponds to taking the union of links separated by a sphere, and as a commutative semigroup it is free. A more interesting semigroup arises if we consider a three-page analogue of tangles.

Let Y be the semigroup presented by 12 generators a_i, b_i, c_i, d_i $(i \in \mathbb{Z}_3)$ and the following 55 relations:

$$\begin{aligned}
a_i &= a_{i+1} d_{i-1}, \\
b_i &= a_{i-1} c_{i+1}, \\
c_i &= b_{i-1} c_{i+1}, \\
d_i &= a_{i+1} c_{i-1}
\end{aligned} \tag{14}$$

$$d_1 d_2 d_3 = 1, \tag{15}$$

$$b_i d_i = d_i b_i = 1, \tag{16}$$

$$\begin{gathered}
vw = wv, \\
\text{where } v \in \{a_{i-1} b_{i-1},\ d_{i-1} c_{i-1},\ b_{i+1} d_{i-1} d_{i+1} b_{i-1}\},\ w \in \{a_i,\ b_i,\ c_i,\ b_{i-1} d_i d_{i-1}\},
\end{gathered} \tag{17}$$

$i \in \mathbb{Z}_3$ (one of the relations (16) may be skipped since it can be deduced from the others, so, actually, we have 54 relations).

For a word $w \in W$ we denote by $[w]$ the equivalence class of w in Y. For a given three-page link diagram L we denote by w_L the word that encodes L.

Theorem 3 1) *Two three-page link diagrams L_1 and L_2 represent equivalent unordered nonoriented links in $\mathbb{R}^3$ if and only if the relation $[w_{L_1}] = [w_{L_2}]$ holds in the semigroup Y.*
2) *An element $x \in Y$ has the form $[w_L]$, where L is a three-page link diagram, if and only if the element x is central.*

In other words, the map
$$L \longmapsto [w_L]$$
induces a one-to-one correspondence between the isotopy classes of unordered nonoriented links and central elements of the semigroup Y.

Proof. To begin with, we generalize the notion of three-page link diagram.

Let Π be a plane perpendicular to the binding line l, let r_1, r_2, r_3 be the three rays $r_i = \Pi \cap P_i$, and let O be the intersection point of Π with the binding line, $O = \Pi \cap l$. On each ray r_i we mark an infinitely many equidistant points:
$$E_{i,k} \in r_i, \quad i \in \mathbb{Z}_3, \quad k \in \mathbb{N},$$
$$\overrightarrow{OE}_{i,1} = \overrightarrow{E_{i,k}E}_{i,k+1} \quad \forall k \in \mathbb{N}.$$
We denote by E the union of all three series of points, $E = \{E_{i,k}\}$, $i \in \mathbb{Z}_3, k \in \mathbb{N}$.

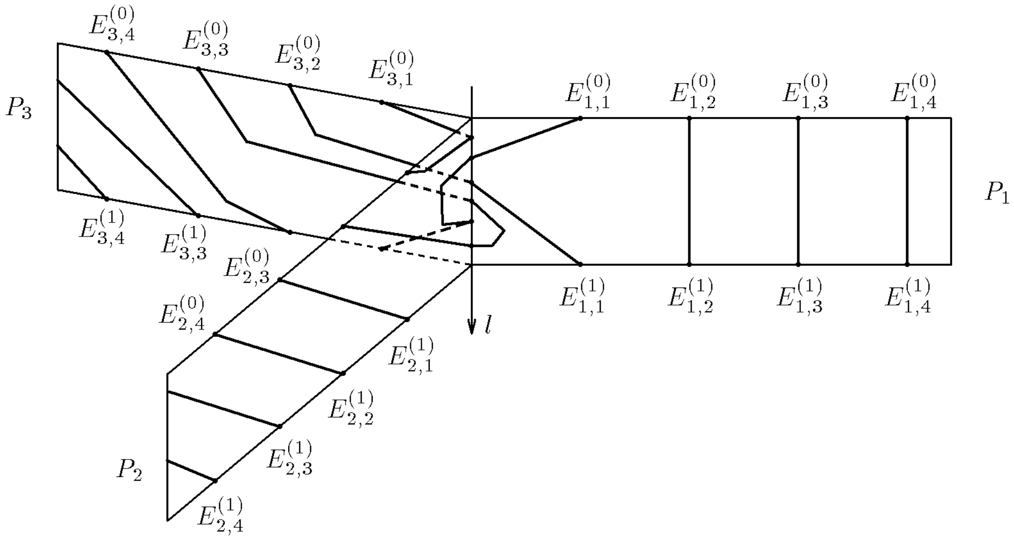
Using the orthogonal projections we represent Euclidean space $\mathbb{R}^3$ as the direct product $\Pi \times l$ and identify l with the real line $\mathbb{R}$. We denote the point sets $E \times 0$ and $E \times 1$ by $E^{(0)} = \{E^{(0)}_{i,k}\}$ and $E^{(1)} = \{E^{(1)}_{i,k}\}$, respectively. The planes $\Pi \times 0$ and $\Pi \times 1$ are denoted by $\Pi^{(0)}$ and $\Pi^{(1)}$.

Definition 4. By a *three-page tangle* we mean a polygonal line $T \subset \mathbb{Y} \cap (\Pi \times [0, 1])$ such that
1) $T \cap (\Pi^{(0)} \cup \Pi^{(1)}) = \partial T = E^{(0)} \cup E^{(1)}$;
2) T is transversal to the binding line l (see Definition 1);
3) the orthogonal projection of any component of $T \cap P_i$ to l is a monotone function for all $i \in \mathbb{Z}_3$.

Two three-page tangles are said to be *equivalent* if they are (ambient) isotopic in $\Pi \times [0, 1]$ (the boundary ∂T is assumed to be fixed during the isotopy).

We recall that in this paper by a plygonal line we mean a one-dimensional PL-subvariety of $\mathbb{R}^3$ each of whose components is compact. An example of a three-page tangle is shown in Fig. 9.

Figure 9: A three-page tangle



Note that for any three-page tangle T there are integers n_1, n_2, n_3, N such that the point $E^{(0)}_{i,k}$ is connected with $E^{(1)}_{i,k+n_i}$ by a component of T lying entirely in P_i for all $i \in \mathbb{Z}_3$, $k > N$. In each isotopy class of three-page tangles one can find a representative T almost all of whose components are straight line segments. Moreover, if T is a polygonal line embedded in $\Pi \times [0,1]$ with the boundary $\partial T = E^{(0)} \cup E^{(1)}$ such that, for some $n_1, n_2, n_3, N \in \mathbb{Z}$, we have $[E^{(0)}_{i,k}, E^{(1)}_{i,k+n_i}] \subset T$ for all $i \in \mathbb{Z}_3$, $k > N$, then T is isotopic (in $\Pi \times [0,1]$) to a three-page tangle. Here $[Q, R]$ denotes the straight line segment connecting the points Q and R.

The composition of three-page tangles is defined as for ordinary tangles and braids. Namely, let τ_1, τ_2 be the following affine transforms of $\mathbb{R}^3$:

$$\tau_1(Q,t) = (Q, t/2), \quad \tau_2(Q,t) = (Q, (t+1)/2), \quad Q \in \Pi,\ t \in \mathbb{R}.$$

The *composition* $T_1 \circ T_2$ of two three-page tangles is defined as the union

$$\tau_1(T_1) \cup \tau_2(T_2).$$

Endowed with this operation, the set of isotopy classes of three-page tangles becomes a monoid. The identity element if the isotopy class of the three-page tangle

$$I = E \times [0,1].$$

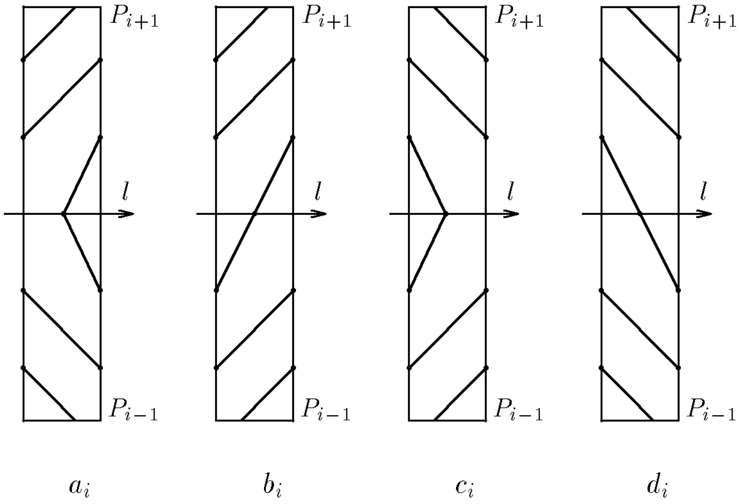
We claim that the semigroup of isotopy classes of three-page tangles is isomorphic to Y. We now construct this isomorphism.

We again denote the vertices of T lying on the binding line l by $A_1, \ldots, A_{m(T)}$.

Similarly to the case of three-page link diagrams, any three-page tangle T is encoded by a word w_T in the alphabet $\mathbb{A}$. It is easy to see that now an arbitrary word in the alphabet $\mathbb{A}$ encodes a three-page tangle and the corresponding three-page tangle can be restored from the word uniquely up to an ambient isotopy in $\mathbb{Y}$. We recall that in the case of links only completely balanced words were allowed. An analog of the second assertion of Proposition 1 holds for three-page tangles: three-page tangles T and T' are ambient isotopic in $\mathbb{Y} \cap (\Pi \times [0,1])$ if and only if the words w_T and$w_{T'}$ coincide.

The generators a_i, b_i, c_i, d_i of the semigroup of three-page tangles are shown in Fig. 10. The rest of the

Figure 10: Geometric representation of the generators of the semigroup Y.



three-page tangles, which is not indicated in the pictures, is of the form $E_i \times [0,1]$, where $E_i = \{E_{i,k}\}_{k>0}$.

Drawing the corresponding picutres, we can readily see that relations (14–17) hold in the semigroup of three-page tangles. We must prove that these relations are sufficient to realize the equivalence of three-page tangles. The proof of this fact essentially repeats that of Theorem 2. We only comment on differences between the relations (14–17) and the relations (1–3).

First of all, in the semigroup of three-page links, we did not write out the relations $b_i d_i = d_i b_i = 1$ because the words $b_i d_i$ and $d_i b_i$ are not balanced. Clearly, in the semigroup of three-page tangles the elements b_i and d_i are inverse to each other. Therefore, the relations

$$a_i = a_{i+1} d_{i-1}, \quad \text{and} \quad c_i = b_{i-1} c_{i+1}$$

in (14) are equivalent to

$$a_i = a_{i-1} b_{i+1}, \quad \text{and} \quad c_i = d_{i+1} c_{i-1},$$

respectively. The relations

$$b_i = d_{i+1} d_{i-1}, \quad \text{and} \quad d_i = b_{i-1} b_{i+1}$$

are equivalent to (15).

Finally, in comparison with (17) to (2–3), we see that we have omitted the relations of the form $vw = wv$, where v is the same as in (17), and w is one of the following:

$$b_{i-1} a_i d_{i-1}, \quad b_{i-1} b_i d_{i-1}, \quad b_{i-1} c_i d_{i-1}. \tag{18}$$

In the semigroup Y we have

$$\begin{aligned} b_{i-1} a_i d_{i-1} &= b_{i-1} a_{i+1} d_{i-1} d_{i-1} \\ &= b_{i-1} a_{i+1} b_{i+1} b_i d_{i-1} \\ &= a_{i+1} b_{i+1} b_{i-1} b_i d_{i-1} \\ &= a_{i+1} d_{i-1} d_i b_{i-1} b_i d_{i-1} \\ &= a_i d_i \; b_{i-1} b_i d_{i-1}, \end{aligned}$$

$$b_{i-1} b_i d_{i-1} = (b_{i-1} d_i d_{i-1})^{-1},$$

$$\begin{aligned} b_{i-1} c_i d_{i-1} &= b_{i-1} b_{i-1} c_{i+1} d_{i-1} \\ &= b_{i-1} d_i d_{i+1} c_{i+1} d_{i-1} \\ &= b_{i-1} d_i d_{i-1} d_{i+1} c_{i+1} \\ &= b_{i-1} d_i d_{i-1} b_i b_{i-1} c_{i+1} \\ &= b_{i-1} d_i d_{i-1} \; b_i c_i. \end{aligned}$$

Thus, if $v \in Y$ commutes with $a_i, b_i, c_i, d_i, b_{i-1} d_i d_{i-1}$ then it commutes with the elements (18) as well.

From now on, we identify the semigroup Y with the semigroup of isotopy classes of three-page tangles. To complete the proof of Theorem 3, we must describe the center of the semigroup Y. We need the following definition.

Definition 5. We say that a three-page tangle T is *separated* if there is a three-dimensional ball $\mathcal{B} \subset \Pi \times [0,1]$ such that

1) the boundary $\partial\mathcal{B} \cong S^2$ is disjoint with T (thus, the intersection $T \cap \mathcal{B}$ consists only of closed components of T);

2) the remaining part $T - (T \cap \mathcal{B})$ is a three-page tangle equivalent to the identity tangle $I = E \times [0,1]$.

It is obvious that a three-page tangle equivalent to a separated tangle is also separated. Thus, we can speak of the separated elements of the semigroup Y. Each separated three-page tangle T defines a link in $\mathbb{R}^3$ which is the intersection of the corresponding ball $\mathcal{B}$ with T.

The following two statements are obvious.

Lemma 10 *An element of Y is separated if and only if it can be represented by a completely balanced word in the alphabet $\mathbb{A}$.*

Lemma 11 *Let w be a completely balanced word. Then an element $x \in Y$ is separated if and only if the element $x \cdot [w]$ is separated.*

The following lemma completes the proof of Theorem 3.

Lemma 12 *An element of Y is separated if and only if it belongs to the center of Y.*

PROOF. The only nonobvious fact is the "if" part.

Let a three-page tangle T represents a central element $x = [x_1 \dots x_n] \in Y$, $x_1, \dots, x_n \in \mathbb{A}$. From $b_i^N a_i d_i^N x = x b_i^N a_i d_i^N$ it follows that for a sufficiently large k and any $i \in \mathbb{Z}_3$ the three-page tangle T has a strand joining $E_{i,k}^{(0)}$ and $E_{i,k}^{(1)}$ and not intersecting the binding line. It readily follows from this that $x_1 + \dots + x_n = 0 \; \forall i \in \mathbb{Z}_3$.

Therefore, for a sufficiently large N the word $a_1^N a_2^N w_T c_2^N c_1^N$ is completely balanced. At the same time $[a_1^N a_2^N w_T c_2^N c_1^N] = [w_T a_1^N a_2^N c_2^N c_1^N]$, where the word $a_1^N a_2^N c_2^N c_1^N$ is completely balanced. It follows from Lemma 11 that the element $x = [w_T]$ is separated. □

Remark 2. Note that in the semigroup Y, we have the following relations:

$$a_{i+1} c_{i-1} d_i = 1 = d_i a_{i+1} c_{i-1},$$

which mean that the elements a_i have the right inverses and the elements c_i have the left inverses. However, the following inequility holds

$$c_{i-1} d_i a_{i+1} \neq 1 \neq c_{i-1} d_i a_{i+1}.$$

Moreover, the factorsemigroup $Y/(c_1 d_2 a_3)$ is isomorphic to $\mathbb{Z}^3$. The isomorphism is defined by the inclusion $\mathbb{A} \subset \mathbb{Z}^3$. The image of Y is not the entire group $\mathbb{Z}^3$ but the subgroup

$$\{(n_1, n_2, n_3) \in \mathbb{Z}^3 | \; n_1 + n_2 + n_3 \equiv 0 \, (\mathrm{mod}\, 2)\} \cong \mathbb{Z}^3$$

(this is a face-central-cubic lattice). Consequently, **there exists no non-Abelian representation $\rho : Y \to G$, where G is a group.**

Now we proceed with the study of braid-like elements of the subgroup Y.

Theorem 4 *Let D be the abstract group presented by generators d_1, d_2, d_3 and seven relations*

$$d_1 d_2 d_3 = 1, \tag{19}$$

$$d_i u_i = u_i d_i, \tag{20}$$

$$u_i u_{i+1} = u_{i+1} u_i, \tag{21}$$

where $u_i = d_{i+1}^{-1} d_{i-1} d_{i+1} d_{i-1}^{-1}$, $i \in \mathbb{Z}_3$. Let D_i be the subgroup of D generated by d_i and u_{i+1} and let $\widehat{D}_i$ be the intersection $D_{i+1} \cap D_{i-1}$, $i \in \mathbb{Z}_3$. Then

1) D is isomorphic to the group of invertible elements of the semigroup Y, and an isomorphism is defined by identifying the generators d_i of D with the corresponding generators of Y;

2) the commutator subgroup

$$D' = \{xyx^{-1}y^{-1} \,|\, x, y \in D\}$$

is isomorphic to the braid group B_∞ on infinitely many strands, and we have

$$D/D' = \mathbb{Z}^2;$$

3) the subgroups D_i and $\widehat{D}_i$ are centralizers of each other for any $i \in \mathbb{Z}_3$;

4) the subgroup $\widehat{D}_i$ is isomorphic to the braid group B_∞ for any $i \in \mathbb{Z}_3$;

5) the subgroup D_i is isomorphic to a semidirect product $B_\infty \leftthreetimes \mathbb{Z}$ for any $i \in \mathbb{Z}_3$.

Remark 3. If $d_3 = d_2^{-1} d_1^{-1}$, then the relations (20) and (21) are equivalent to the relations

$$d_1 d_3 d_1^{-1} d_2 d_1 = d_2 d_1 d_2^{-1} d_3 d_2 = d_3 d_2 d_3^{-1} d_1 d_3, \tag{22}$$

$$u_3 u_2 = u_2 u_3. \tag{23}$$

Thus, the number of generators and defining relations in the presentation of the group D can be reduced to two and three, respectively. Another way of writing relations (22) and (23) is given in the Abstract, where we set $a = g_1^{-1}$, $b = g_2$.

Proof. To begin with, we recall well-known facts about braid groups. Let F be an arbitrary locally finite subset of the plane Π and let B_F be the group whose elements are the isotopy classes of braids, i.e. polygonal lines $T \subset \Pi \times [0,1]$ such that
1) $\partial T = (F \times \{0\}) \cup (F \times \{1\})$;
2) the restriction of the projection $\Pi \times [0,1] \to [0,1]$ to any connected component (strand) of T is a monotone function;
3) all strands of T except for finitely many of them are straight line segments parallel to the line l.

If there is a bijection between two locally finite subsets $F_1, F_2 \subset \Pi$, then the groups B_{F_1} and B_{F_2} are isomorphic, an isomorphism is defined by any homeomorphism $h : \Pi \to \Pi$ such that $h(F_1) = F_2$. In particular, if F is an n-point set, then the group B_F is isomorphic to the Artin braid group B_n on n strands. If the set F is infinite, then the group B_F is isomorphic to the braid group B_∞ on infinitely many strands, which is the direct limit of the sequence

$$B_1 \subset B_2 \subset B_3 \subset \dots.$$

There are two presentations of the braid group 1 B_∞ by generators and relations, the $\mathbb{Z}$-presentation and the $\mathbb{N}$-presentation. Both groups $B_\mathbb{Z}$ and $B_\mathbb{N}$ are generated by elements σ_k, where $k \in \mathbb{Z}$ for the group $B_\mathbb{Z}$ and $k \in \mathbb{N}$ for the group $B_\mathbb{N}$ with the defining relations:

$$\sigma_j \sigma_k = \sigma_k \sigma_j, \quad k > j+1, \tag{24}$$

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \tag{25}$$

where $j, k \in \mathbb{Z}$ or $j, k \in \mathbb{N}$, respectively.

Let H be the group of invertible elements of the semigroup Y. A three-page tangle T represents an invertible element of Y if and only if there is an isotopy that preserves the boundary ∂T and transforms T to a polygonal line $T' \subset \Pi \times [0,1]$ such that
1) the restriction of the projection $\Pi \times [0,1] \to [0,1]$ to any connected component of T' is a monotone function and
2) for some $n_1, n_2, n_3 \in \mathbb{Z}$ we have $[E^{(0)}_{i,k}, E^{(1)}_{i,k+n_i}] \subset T'$ for all $i \in \mathbb{Z}_3$ and for all sufficiently large k.

Recall (see Section 3) that the alphabet $\mathbb{A}$ is identified with a subset of $\mathbb{Z}^3$. This allows one to define a mapping $\varphi : W \to \mathbb{Z}^3$ by the formula $\varphi(x_1 \dots x_n) = x_1 + \dots + x_n$, where $x_k \in \mathbb{A}$. This mapping can be factorized to a semigroup homomorphism $Y \to \mathbb{Z}^3$, which we again denote by φ. The triple (n_1, n_2, n_3) mentioned above is the value of φ at the element $[w_T] \in Y$. The image of $\varphi\big|_H$ is isomorphic to $\mathbb{Z}^2$ and consists of all vectors $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that

$$n_1 + n_2 + n_3 = 0.$$

The kernel $\ker(\varphi\big|_H)$ is isomorphic to the braid group $B_E \cong B\infty$, and we have $H/B_E = \mathbb{Z}^2$.

We are now going to prove that $D \cong H$. Relations (19–21) hold in the group H (in fact, they are just the relations in (14–17) that contain only the letters b_i, d_i). Thus, we have a homomorphism $\eta : D \to H$ that sends the generators d_i of D to the corresponding elements d_i of H.

It trivially follows from relations (19–21) that

$$D/D' \cong \mathbb{Z}^2 \quad \text{and} \quad \eta(D') \subset B_E.$$

Thus, it suffices to show that the restriction $\eta\big|_{D'} : D' \to B_E$ is an isomorphism.

Surjectivity of the mapping $\eta : D' \to B_E$.
We must show that any element of B_E is isotopic in $\Pi \times [0,1]$ to a *three-page braid*, i.e. a three-page tangle T such that the word w_T contains only the letters b_i and d_i.

Let $\mathcal{W}$ (wall) be a plane in $\mathbb{R}^3$ parallel to the bindning line l and separating the set E_1 from $E_2 \cup E_3$. Any element of B_{E_1} can be presented by a braid not intersecting the plane $\mathcal{W}$. This defines an inclusion $B_{E_1} \hookrightarrow B_E$.

For any element $x \in B_E \subset H$ there is an integer N such that

$$(b_2 d_3)^N \, x \, (b_3 d_2)^N \in B_{E_1}. \tag{26}$$

Therefore, it suffices to show that $\eta(D') \supset B_{E_1}$.

The group B_{E_1} is canonically isomorphic to the group $B_{\mathbb{N}}$. The standard generators σ_k of $B_{\mathbb{N}}$ can be realized in the semigroup Y as follows:

$$\sigma_k = d_3^{k-1}\, b_2 d_3 d_2 b_3 \, b_3^{k-1} = \eta(d_3^{k-1} u_1 d_3^{1-k}), \qquad k > 0. \tag{27}$$

We see that $\sigma_k \in \eta(D')$, and hence $B_{E_1} \subset \eta(D')$.

Injectivity of the mapping $\eta : D' \to B_E$.
By (26) and (27), it suffices to show that the elements

$$\overline{\sigma}_k = d_3^{k-1} u_1 d_3^{1-k} \tag{28}$$

of the group D satisfy relations (24) and (25). We begin with relation (25). We have

$$\begin{aligned}
d_3^2 u_2 d_3^{-2} &= d_3^2 (d_3^{-1} d_1 d_3 d_1^{-1}) d_3^{-2} \\
&= d_3 d_1 d_3 d_1^{-1} d_3^{-2} \\
&= d_2^{-1} d_3 d_2 d_3^{-1} \\
&= u_1,
\end{aligned}$$

$$\begin{aligned}
\overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1 (\overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2)^{-1} &= u_1 d_3 u_1 d_3^{-1} u_1 d_3 u_1^{-1} d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 u_1 d_3^{-1} (d_2^{-1} d_3 d_2 d_3^{-1}) d_3 u_1^{-1} d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 u_1 d_1 d_3 d_2 u_1^{-1} d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 u_1 d_1 d_3 d_2 (d_3 d_2^{-1} d_3^{-1} d_2) d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 u_1 d_1 d_3 d_1^{-1} d_2^{-1} d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 (d_2^{-1} d_3 d_2 d_3^{-1}) d_1 d_3 d_1^{-1} d_2^{-1} d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 d_2^{-1} d_3 d_2 u_2 d_2^{-1} d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= u_1 d_3 d_2^{-1} d_3 u_2 d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= (d_2^{-1} d_3 d_2 d_3^{-1}) d_3 d_2^{-1} d_3 u_2 d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= d_2^{-1} d_3^2 u_2 d_3^{-1} d_2 d_3^{-1} u_1^{-1} d_3 u_1^{-1} d_3^{-1} \\
&= d_2^{-1} d_3^2 u_2 d_3^{-1} d_2 d_3^{-1} (d_3 d_2^{-1} d_3^{-1} d_2) d_3 u_1^{-1} d_3^{-1} \\
&= d_2^{-1} d_3^2 u_2 d_3^{-2} d_2 d_3 u_1^{-1} d_3^{-1} \\
&= d_2^{-1} u_1 d_2 d_3 u_1^{-1} d_3^{-1} \\
&= d_2^{-1} u_1 d_1^{-1} u_1^{-1} d_3^{-1} \\
&= d_2^{-1} d_1^{-1} d_3^{-1} \\
&= 1,
\end{aligned}$$

$$\overline{\sigma}_k \overline{\sigma}_{k+1} \overline{\sigma}_k (\overline{\sigma}_{k+1} \overline{\sigma}_k \overline{\sigma}_{k+1})^{-1} = d_3^{k-1} \overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1 (\overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2)^{-1} d_3^{1-k} = 1.$$

Now, we shall prove by induction that, for $k > 0$, the following relations hold in D:

$$d_3^k u_1 d_3^{-k} = d_2^{-k} u_1 d_2^k, \tag{29}$$

$$u_1(d_3^{k+1} u_1 d_3^{-k-1}) = (d_3^{k+1} u_1 d_3^{-k-1}) u_1. \tag{30}$$

For $k = 1$, relation (30) follows from (21) and the relation

$$d_3^2 u_2 d_3^{-2} = u_1, \tag{31}$$

which has been proved above. Indeed,

$$u_1(d_3^2 u_1 d_3^{-2}) = d_3^2 u_2 u_1 d_3^{-2} = d_3^2 u_1 u_2 d_3^{-2} = (d_3^2 u_1 d_3^{-2}) u_1.$$

As for (29), in the case of $k = 1$ we have

$$d_3 u_1 d_3^{-1} = d_2^{-1} d_1^{-1} u_1 d_1 d_2 = d_2^{-1} u_1 d_2.$$

Assume that (29) and (30) hold for $0 < k \leq n-1$. Then

$$\begin{array}{rcl} d_3^n u_1 d_3^{-n} & = & (u_1 d_3)^{n-1} d_3 u_1 d_3^{-1} (u_1 d_3)^{1-n} \\ & = & (d_2^{-1} d_3 d_2)^{n-1} d_2^{-1} u_1 d_2 (d_2^{-1} d_3 d_2)^{1-n} \\ & = & d_2^{-1} d_3^{n-1} u_1 d_3^{1-n} d_2 \\ & = & d_2^{-n} u_1 d_2^n. \end{array}$$

The first equality is obtained by applying (30) $n-1$ times. Further,

$$\begin{array}{rcl} u_1(d_3^{n+1} u_1 d_3^{-n-1}) & = & u_1 d_3^2 (d_2^{1-n} u_1 d_2^{n-1}) d_3^{-2} \\ & = & d_3^2 u_2 (d_2^{1-n} u_1 d_2^{n-1}) d_3^{-2} \\ & = & d_3^2 (d_2^{1-n} u_1 d_2^{n-1}) u_2 d_3^{-2} \\ & = & d_3^2 (d_2^{1-n} u_1 d_2^{n-1}) d_3^{-2} u_1 \\ & = & (d_3^{n+1} u_1 d_3^{-n-1}) u_1 \end{array}$$

It follows from (30) that

$$\begin{array}{rcl} \overline{\sigma}_j \overline{\sigma}_k & = & d_3^{j-1} u_1 (d_3^{k-j} u_1 d_3^{j-k}) d_3^{1-j} \\ & = & d_3^{j-1} (d_3^{k-j} u_1 d_3^{j-k}) u_1 d_3^{1-j} \\ & = & \overline{\sigma}_k \overline{\sigma}_j \end{array}$$

for $k > j+1$. This completes the proof of parts 1) and 2) of Theorem 4.

Note that it follows from our considerations that all the elements (28), $k \in \mathbb{Z}$, satisfy relations (24) and (25). These elements generate the subgroup $D_3 \cap D'$. Thus, we have an embedding

$$B_{E_1 \cup E_2} \hookrightarrow D_3.$$

In D_3, there is also a subgroup isomorphic to $\mathbb{Z}$ which is generated by d_3.

Any element x of D_3 is of the form $x = y d_3^k$, where $y \in B_{E_1 \cup E_2}$. This means that the group D_3 is isomorphic to the semidirect product

$$D_3 \cong B_{E_1 \cup E_2} \underset{\xi}{\rtimes} \mathbb{Z},$$

where the action ξ of $\mathbb{Z}$ on $B_{E_1 \cup E_2}$ is defined by the formula

$$\xi(d_3)(\overline{\sigma}_k) = \overline{\sigma}_{k-1}.$$

We can readily see that the subgroup $\widehat{D}_i$ coincides with B_{E_i} for $i \in \mathbb{Z}_3$. We leave the proof of part 3) of Theorem 4 to the reader. □

Remark 4. The isomorphism $B_{\mathbb{Z}} \xrightarrow{\cong} D'$ can be represented explicitly as follows:

$$\sigma_k \mapsto \begin{cases} d_3^{k-1} u_1 d_3^{1-k}, & \text{for } k \geq 1, \\ d_2 (d_2^{-1} d_3)^{k/2} u_1 (d_2^{-1} d_3)^{-k/2} d_2^{-1}, & \text{for even } k,\ k \leq 0, \\ (d_2^{-1} d_3)^{(k-1)/2} u_1 (d_2^{-1} d_3)^{(1-k)/2}, & \text{for odd } k,\ k \leq -1 \text{ is odd,} \end{cases}$$

In conclusion of this section we indicate an algebraic property of the group D that readily follows from the geometric realization and is characteristic in a certain sence.

Proposition 3 *For any $k > 0$ and $l \geq 0$, there is a monomorphism (which is obviously unique) $\psi_{k,l} : D \to D$ such that*

$$\psi_{k,l}(d_1) = d_2^l d_1^k d_2^{-l}, \qquad \psi_{k,l}(d_2) = d_1^{-l} d_2^k d_1^l. \tag{32}$$

The homomorphisms $\psi_{k,l}$ satisfy the relations

$$\psi_{k,l} \circ \psi_{k',l'} = \psi_{kk',kl'+l}, \tag{33}$$

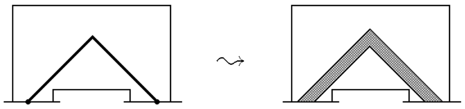
which, in turn, imply the defining relations (19–21) *defining the group D.*

6 Band-links

In this section we apply our three-page technique to band-links. By a band-link we mean a two-dimensional compact PL-submanifold $R \subset \mathbb{R}^3$ (with boundary) each of whose connected components is homeomorphic to either the ring $S^1 \times [0,1]$ or the Möbius band $S^1 \ltimes [0,1]$. The notion of band-link is very close to that of framed link. Namely, if we allow only oriented components, i.e. components homeomorphic to the ring $S^1 \times [0,1]$, then the isotopy classes of such band-links are in a natural one-to-one correspondence with the isotopy classes of framed links. For our purposes, it is more convenient to admit nonoriented components. Two band-links R_1, R_2 are said to be *equivalent* (in which case we write $R_1 \sim R_2$) if they are isotopic in $\mathbb{R}^3$.

There is an obvious way to transform any three-page link diagram into a band-link. To this end, one must replace each edge by its small neighborhood in the corresponding half-plane P_i (see Fig. 11). The

Figure 11: Transforming a three-page diagram into a band-link.



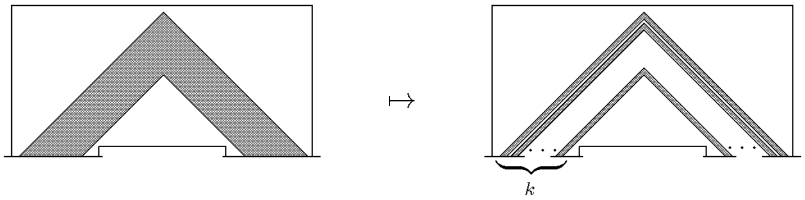
band-link obtained in this way from a three-page link L is denoted by $\check{L}$.

For band-links, we introduce a semigroup $\check{Y}$ similar to the semigroup Y. The semigroup $\check{Y}$ is generated by the same set $\mathbb{A}$ with the defining relations:

$$\begin{aligned} a_i d_i &= a_{i+1} b_{i+1} \\ b_i c_i &= d_{i+1} c_{i+1}, \end{aligned} \tag{34}$$

$$\begin{aligned} b_i b_{i-1} b_{i+1} &= a_{i-1} c_{i+1} d_i \\ d_i d_{i+1} d_{i-1} &= a_{i+1} c_{i-1} b_i, \end{aligned} \tag{35}$$

Figure 12: Geometric realization of the homomorphism ψ_k.



$$b_i d_i = d_i b_i = 1, \tag{36}$$

$$a_i b_i c_i = b_i, \tag{37}$$

$$\begin{gathered} vw = wv, \\ \text{where } v \in \{a_{i-1}b_{i-1},\ d_{i-1}c_{i-1},\ b_{i+1}d_{i-1}d_{i+1}b_{i-1},\ d_{i-1}d_i d_{i+1}\}, \\ w \in \{a_i,\ b_i,\ c_i,\ b_{i-1}d_i d_{i-1},\ d_{i+1}d_{i-1}\}, \qquad i \in \mathbb{Z}_3. \end{gathered} \tag{38}$$

For a word $w \in W$ we denote the equivalence class of w in $\check{Y}$ by $[\![w]\!]$.

Theorem 5 *1) For any band-link R, there is a three-page link diagram L such that $R \sim \check{L}$;*
2) for two three-page links L_1, L_2 the band-links $\check{L}_1$ and $\check{L}_2$ are equivalent if and only if $[\![w_{L_1}]\!] = [\![w_{L_2}]\!]$;
3) an element $x \in \check{Y}$ is of the form $x = [\![w_L]\!]$, where L is a three-page link diagram, if and only if x is a central element.

The geometric realization of the semigroup $\check{Y}$ is almost the same as that of the semigroup Y. An element of $\check{Y}$ can be realized as an isotopy classe of three-page tangles whose strands are replaced by bands. The proof of the last theorem differs from that of Theorem 3 only in details which we leave to the reader.

The following two assertions are easy consequences of the geometric realization of the semigroups Y and $\check{Y}$.

Proposition 4 *There is an epimorphism $\theta : \check{Y} \to Y$ such that $\theta([\![w]\!]) = [w]$ for all $w \in W$. The kernel* $\ker(\theta)$ *is isomorphic to the free Abelian group $\mathbb{Z}^E_{\mathrm{fin}}$ generated by the set E. The isomorphism sends $E_{i,k}$, where $i \in \mathbb{Z}_3$ and $k \in \mathbb{N}$, to $[\![d^{k-1}_{i-1} b_{i+1} b_i b^k_{i-1}]\!]$.*

Proposition 5 *For any $k > 0$ there exists a homomorphism $\psi_k : \check{Y} \to \check{Y}$ such that*

$$\psi_k(a_i) = a_i^k, \quad \psi_k(b_i) = b_i^k, \quad \psi_k(c_i) = c_i^k, \quad \psi_k(d_i) = d_i^k, \qquad i \in \mathbb{Z}_3.$$

Geometrically, the homomorphisms ψ_k are interpreted as the local cutting of each band into k narrower bands as shown in Fig. 12.

Now we shall translate classical results by L. Kauffman, who constructed polynomial invariants for framed nonoriented links. Recall that Kauffman's bracket polynomial [10] is defined by the skein relation

$$\langle\ \ \rangle(A) - A \cdot \langle\ \ \rangle(A) - A^{-1} \cdot \langle\ \ \rangle(A) = 0 \tag{39}$$

and the normalizing condition

$$\langle\ \ \rangle = 1. \tag{40}$$

The bracket polynomial satisfies the relation

$$\Big\langle \quad \Big\rangle(A) = -A^3 \cdot \Big\langle \quad \Big\rangle(A). \tag{41}$$

We can extend the bracket polynomial to a band-link invariant by setting

$$\Big\langle \quad \Big\rangle(A) = \sqrt{-A^3} \cdot \Big\langle \quad \Big\rangle(A). \tag{42}$$

Let $\mathbb{Z}[\check{Y}]$ be the semigroup algebra of the semigroup $\check{Y}$, let $\mathbb{Z}[\check{Y}][z, z^{-1}]$ be the algebra of Laurent polynomials in variable z with coefficients in $\mathbb{Z}[\check{Y}]$, and $K_{\langle\rangle}$ be the quotient algebra

$$\mathbb{Z}[\check{Y}][z, z^{-1}]/(b_3b_2b_1 - z^3, b_3b_1 + d_2z + c_1a_3z^2). \tag{43}$$

The following statement is the three-page version of Kauffman's result [10].

Proposition 6 *1) The canonical homomorphism $\mathbb{Z}[z, z^{-1}] \to K_{\langle\rangle}$ is injective.*
2) For any three-page link diagram L, the following relation holds in the algebra $K_{\langle\rangle}$:

$$[\![w_L]\!] = -(z^4 + z^{-4}) \cdot \big\langle \check{L} \big\rangle(-z^2). \tag{44}$$

A similar algebra can be constructed for Kauffman's L-polynomial $\mathcal{L}(z, \alpha)$, which is defined by the skein relations

$$\mathcal{L}(z, \alpha) + \mathcal{L}(z, \alpha) = z \cdot \Big(\mathcal{L}(z, \alpha) + \mathcal{L}(z, \alpha)\Big), \tag{45}$$

$$\mathcal{L}\;(z, \alpha) = \sqrt{\alpha} \cdot \mathcal{L}\;(z, \alpha) \tag{46}$$

and the normalizing condition

$$\mathcal{L} = 1. \tag{47}$$

Let $K_{\mathcal{L}}$ be the algebra

$$\mathbb{Z}[\check{Y}][z, z^{-1}, t, t^{-1}]/(b_3b_2b_1 - t,\ b_3b_1d_3d_1 + b_1b_3d_1d_3 - c_1a_1z - z). \tag{48}$$

Kauffman's theorem [11, Theorem 2.3] on the existence and uniqueness of the L-polynomial translates into the following

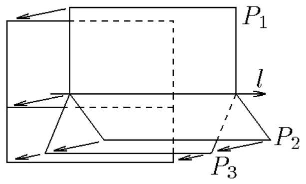
Proposition 7 *1) The canonical homomorphism $\mathbb{Z}[z, z^{-1}, t, t^{-1}] \to K_{\mathcal{L}}$ is injective.*
2) For any three-page link diagram L the following relation holds in the algebra $K_{\mathcal{L}}$:

$$[\![w_L]\!] = ((t^2 + t^{-2})/z - 1) \cdot \mathcal{L}_{\check{L}}(z, t^2). \tag{49}$$

7 An algorithm for transforming a planar diagram into a three-page diagram.

In this section we describe a converting procedure that is easily programmed. This procedure is used as the first part of partial untangling algorithms proposed here. It receives as the initial data a combinatorial description of a planar diagram (Gauss code) and converts these data to a combinatorial description of a three-page link diagram representing the same link.

Figure 13: Projecting $\mathbb{Y}$ to the plane.



To represent graphically steps of the algorithm we project the three-page book $\mathbb{Y}$ to the plane so that the page P_1 is mapped to the upper half-plane and the pages P_2 and P_3 are mapped to the lower one (fiture 13). The projection of edges lying in P_3 will be depicted by double line $=$. The projections of all other edges will be depicted by bold line.

Initially, we have a combinatorial description of a planar link diagram. This means that we are given a 4-valent graph with an additional structure at nodes. We assume that the graph is embedded into the upper half-plane. We shall successively resolve crossings of the diagram by adding vertices at the horizontal line, which is the projection of the binding line l, and connecting them by edges of two types in the lower half-plane.

The exact rules are the following. First, we cut an edge of the graph, attach the obtained ends to two new vertices at the horizontal line and connect them by an edge in the lower half-plane. We put a special mark at the right added vertex (Fig. 14).

Figure 14: The first step of the converting algorithm.

At each further step of the algorithm, we look at the edge that lies in the upper half-lane and comes from the marked vertex and check if the other end of the edge is attached to a crossing. If so, we resolve the crossing according to the rules shown in Fig. 15. The marked vertex is changed as indicated.

If the marked point is connected with no crossing in the upper half-plane and the marked point is not the rightmost point, then we proceed with the next point (Fig. 16). If the marked point is connected with no crossing and it is the rightmost vertex on the horizontal line, then the procedure stops.

The obtained picture has no crossings in the upper half-plane. This picture can be interpreted in two ways: 1) we may consider it as a planar link diagram, assuming that at the crossings in the lower half-plane the double line is the undercrossing branch and the bold line is the overcrossing one; this diagram is equivalent to the initial one; 2) the obtained picture may be interpreted as the projection of a three-page link diagram, on which the double lines stay for projections of edges lying in P_3 and the bold lines represent the edges lying in P_1 and P_2.

If the initial planar link diagram has n crossings, then the three-page diagram obtained by implementing the above algorithm has no more than $3n + 2$ vertices on the binding line. If we got $3n + 2$ vertices, this always means that the first edge we created in the lower half-plane is trivial and may be contracted. The obtained three-page link diagram will then have $3n + 1$ vertices.

Remark 5. As we have seen above, in order to encode a three-page link diagram we need about $O(\mathrm{m})$ amount of memory, where m is the complexity of the given diagram. However, in practice, our 'untangling technology' uses a 'more explicit' presentation of three-page diagrams rather than words in the alphabet

Figure 15: Resolving crossings

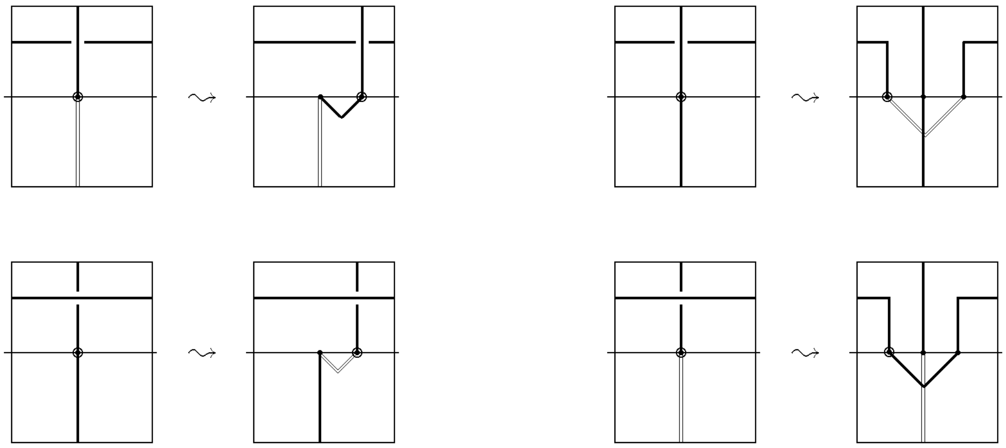
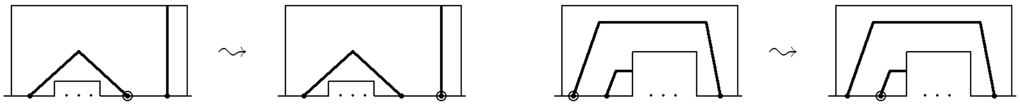


Figure 16:



$\mathbb{A}$. Namely, for each vertex A_i, we permanently keep in memory the references to the vertices that A_i is connected to by edges and in which half-planes those edges lie. This requires $O(m\log(m))$ amount of memory but it is much more convenient for handling the diagram by computer and much faster.

Remark 6. When putting a three-page diagram into computer's memory we consider the points $A_1, \dots, A_m$ up to a cyclic permutation, i.e. we regard A_1 as the next vertex to A_m. Topologically this means, that we consider one-point compactification of $\mathbb{Y}$ and the binding line l is regarded as a circle.

8 Untangling a three-page diagram by simple moves.

Now we shall describe effective procedures, which are realized as computer programs, that take as input a planar knot diagram and *try* to untangle it. This procedures are *not* the complete algorithms for unknot recognition. The result of such a procedure is either the positive answer 'the knot is trivial' or an uncertain answer. The positive answer is given only if the given knot is actually trivial. An important requirement that we postulate for such a procedure is: it must take reasonable time, so that the program can be executed for diagrams having a large number (100–10000) of crossings.

Let L be a three-page link diagram, γ be a connected component of the intersection $L \cap (P_i \cup P_{i+1})$. In Section 2 we defined a simple move as replacing γ by another path $\gamma' \subset (P_i \cup P_{i+1}) - (L - \gamma)$ with the same boundary $\partial\gamma' = \partial\gamma$. We define the *length* of such a path γ' as the number of edges it consists of.

Lemma 13 *There is an algorithm for finding the shortest path γ' such that $\gamma' \subset (P_i \cup P_{i+1}) - (L - \gamma)$ and $\partial\gamma' = \partial\gamma$ that requires no more than $O(m)$ operations, where $m = m(L)$ is the complexity of L.*

Remark 7. The shortest path γ' may not be unique. We assert only that we can find at least one of the shortest paths in $O(m)$, but not necessarily all of them.

Proof. Let $A_{i_1}, \dots, A_{i_k}$ be those vertices of $L \cap l$ that do not belong to γ. The shortest path γ' we are looking for, evidently, can intersect each segment $[A_{i_p}, A_{i_{p+1}}]$ no more than once. Let us choose a point M_p in each interval $(A_{i_p}, A_{i_{p+1}})$ (the point $A_{i_{k+1}}$ is identified with A_{i_1}) in such a way that $\partial\gamma \subset \{M_1, \dots, M_k\}$. Let

$$\partial\gamma = \{M_s, M_t\}.$$

We call a triple (M_p, M_q, j), $j = i$ or $i+1$, *a semi-edge* if the points M_p and M_q can be connected in the half-plane P_j by a path not intersecting $L - \gamma$ (see Remark 8 below for an explanation of this term). We denote a semi-edge of the form (M_p, M_q, i) by $\overset{\frown}{M_p M_q}$ and $(M_p, M_q, i+1)$ by $\underset{\smile}{M_p M_q}$. Note that the order of the points M_p and M_q is essential in this definition.

We inductively define sets $\mathcal{S}_r^\pm$ of semi-edges and sets $\mathcal{M}_r^\pm$ of vertices M_p as follows:

$$\mathcal{S}_1^+ = \{\overset{\frown}{M_p M_s}\}, \qquad \mathcal{S}_1^- = \{\underset{\smile}{M_p M_s}\},$$

$$\mathcal{M}_1^+ = \{M_p \mid (M_p, M_s, i) \text{ is a semi-edge}\},$$

$$\mathcal{M}_1^- = \{M_p \mid (M_p, M_s, i+1) \text{ is a semi-edge}\},$$

and for $r > 1$

$$\mathcal{S}_r^+ = \{\overset{\frown}{M_p M_q} \mid M_q \in \mathcal{M}_{r-1}^-,\ M_p \notin \mathcal{M}_j^\pm \text{ for } j = 1, \dots, r-1\},$$

$$\mathcal{S}_r^- = \{\underset{\smile}{M_p M_q} \mid M_q \in \mathcal{M}_{r-1}^+,\ M_p \notin \mathcal{M}_j^\pm \text{ for } j = 1, \dots, r-1\},$$

$$\mathcal{M}_r^+ = \{M_p \mid \exists\, M_q \text{ such that } \overset{\frown}{M_p M_q} \in \mathcal{S}_r^+\},$$

$$\mathcal{M}_r^- = \{M_p \mid \exists\, M_q \text{ such that } \underset{\smile}{M_p M_q} \in \mathcal{S}_r^-\}.$$

By construction, if the shortest path γ' has length λ, then $M_t \in \mathcal{M}_\lambda^+ \cup \mathcal{M}_\lambda^-$. For example, in the case $M_t \in \mathcal{M}_\lambda^+$, the shortest path γ' is given by a sequence of semi-edges of the form

$$\overset{\frown}{M_t M_{p_{\lambda-1}}} \dots \overset{\frown}{M_{p_2} M_{p_1}}\ \underset{\smile}{M_{p_1} M_s},$$

if λ is even, and of the form

$$\overset{\frown}{M_t M_{p_{\lambda-1}}} \dots \underset{\smile}{M_{p_2} M_{p_1}}\ \overset{\frown}{M_{p_1} M_s},$$

if λ is odd, where $M_{p_j} \in \mathcal{M}_j$.

The number of operations necessary to find all the sets $\mathcal{S}_r^\pm, \mathcal{M}_r^\pm$, $r = 1, \dots, \lambda$ does not exceed

$$C \cdot \sum_{r=1}^{\lambda} (\#\mathcal{M}_r^+ + \#\mathcal{M}_r^-) < 2Cm,$$

where C is some constant. □

Remark 8. As we have already noticed, when storing a three-page diagram in computer's memory, we keep a pair of references for each edge of the diagram: the entry corresponding to the left end of the edge refers to the right end and vise versa. When searching for the shortest path γ', we represent each semi-edge (M_p, M_q, j) by one reference: it is assigned to M_p and points to M_q. Note that for any point M_p and $j = i$ or $i+1$ there is no more than one semi-edge $(M_p, M_q, j) \in \bigcup_r (\mathcal{S}_r^+ \cup \mathcal{S}_r^-)$. Thus, as soon as we get $M_t \in \mathcal{M}_\lambda^\pm$, we can find the shortest path γ' for $O(\lambda)$ operations.

Remark 9. When estimating the number of operations, we does not take into account that, in reality, handling larger arrays and bigger numbers requires more operations for each elementary action like reading from/writing to memory, comparing numbers, etc.

As a corollary to Lemma 13, we have the following statement.

Proposition 8 *For a three-page diagram L of complexity m, an equivalent diagram L_1 such that there is no simple move $L_1 \rightsquigarrow L_2$ with $m(L_2) < m(L_1)$ can be found for $O(m^3)$ operations.*

PROOF. We shall call any connected component of $L \cap (P_i \cup P_{i+1})$, where $i \in \mathbb{Z}_3$, *an elementary path* of the diagram L. Clearly, the number of elementary paths of L that lie in the plane $P_i \cap P_{i+1}$ equals to the number of edges in the half-plane P_{i-1}. Thus the diagram L has exactly $m(L)$ elementary paths.

Therefore, if there exists a simple move $L \rightsquigarrow L'$ satisfying the inequality $m(L') < m(L)$, then one of the simple moves with this property can be found for $\leq Cm^2$ operations. A sequence of diagrams $L, L', L'', \ldots, L^{(k)}$ such that $m(L^{(j)}) < m(L^{(j-1)})$, $j = 1, \ldots, k$, with the diagram $L_1 = L^{(k)}$ unsimplifiable can be found for $< C(m - m_1 + 1)m^2 = O(m^3)$ operations, where $m_1 = m(L_1)$. □

Remark 10. Computer experiments show that the time needed for finding a diagram L_1 mentioned above grows more likely as m^2. There are two possible reasons for that. First, the average length of elementary paths of an arbitrary three-page diagram equals 2 and the average number of semi-edges (M_p, M_q, j) for given M_q and j tends to 2 when $m \to \infty$. This means that the average time for finding the shortest path γ' grows much slower than $O(m)$, since the growth is caused only by increasing of the probability of deviation of the shortest path length and the number of semi-edges with given end from the average value 2 (if the deviation had been uniformly bounded there would have been no growth at all).

The other reason is the following. If the diagram L is far from being optimal, i.e., $m_1 \ll m$, then L seems to have many elementary paths which are not shortest. For such a diagram we usually need to check only few elementary paths in order to find a simpler diagram. On the other hand, if the diagram L is almost optimal, $(m - m_1)/m \ll 1$, then the sequence of diagrams $L', L'', \ldots$ is short.

As a partial algorithm for testing a knot for unknotedness, the procedure described above is not very effective. However, it provides a good base for constructing better algorithms due to the following reasons.

1) Its implementation requires very little computer resources, that is, time and memory.

2) As a combinatorial object, a three-page diagram is extremely simple (about 300 arithmetic and logical operators are enough to realize the converting and simplifying procedures together as a computer program that has no subroutines and function calls). At the same time three-page diagrams represent actual knots in the three-space and simple moves are combinatorial representations of actual isotopies. All this makes the computer program very reliable: a mistake in the realization will much more likely result in halting than in answering 'unknotted' for a nontrivial knot.

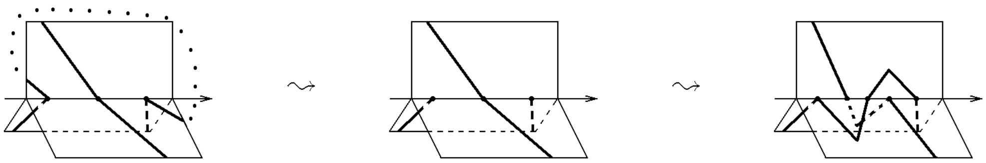
3) The procedure is rather flexible. It admits numerous modifications some of which turn out to be much more effective.

9 Combined moves.

In this section we discuss more complicated three-page link diagram transformations preserving the isotopy class of the link. We call them combined moves.

Before introducing the notion of combined move, let us consider the following example. Suppose, we have a three-page link diagram L such that for some $k \leq m - 2$ the vertices A_k and A_{k+2} are connected by an elementary path γ of length 4 in the plane $P_1 \cup P_2$ and there is no shorter path connecting those points. After removing the path γ we will have a picture like one shown in the center of Fig. 17. If we replace the vertex A_{k+1} by a trivial edge in the half-plane P_3 the points A_k and A_{k+2} could be connected

Figure 17: One-sided combined move.



by an elementary path of length 2 (see the right picture in Fig. 17). As the result we get a three-page diagram L', which is simpler than L:

$$m(L') = m(L) - 4 + 1 + 2 = m(L) - 1.$$

On the other hand, it may happen that an elementary path $\gamma \in P_i \cup P_{i+1}$ of a three-page diagram L can be replaced by another elementary path $\gamma' \in P_i \cup P_{i+1}$, $\partial\gamma' = \partial\gamma$, having the same length as γ, in such a way that some nontrivial edge of L lying in P_{i-1} becomes a trivial edge of the diagram $L' = (L - \gamma) \cup \gamma'$. It means that, by contracting the trivial edge, we can simplify the diagram L'. Note that this contraction can be realized as an isotopy within $\mathbb{Y}$.

We summarize these ideas in the following definition.

Definition 6. We say, that a three-page diagram L' is obtained from L by *a one-sided combined move* if for some $i \in \mathbb{Z}_3$ there are elementary paths $\gamma \subset L \cap (P_i \cup P_{i+1})$, $\gamma' \subset L' \cap (P_i \cup P_{i+1})$ and an isotopy (which is assumed to be continuable to an ambient isotopy in $\mathbb{R}^3$)

$$f : \overline{L - \gamma} \times [0, 1] \to \mathbb{Y}$$

such that

$$f(x, 0) = x \quad \forall x \in \overline{L - \gamma},$$
$$f(x, t) = x \quad \forall x \in \partial\gamma, t \in [0, 1],$$
$$f(\overline{L - \gamma}, 1) = \overline{L' - \gamma'}.$$

The following lemma is obvious.

Lemma 14 *If a three-page diagram L' is obtained from L by a one-sided combined move then the diagrams L and L' define equivalent links in $\mathbb{R}^3$.*

If the elementary path $\gamma \subset L$ is fixed, then finding the simplest diagram L' satisfying Definition 6 is as easy computational problem as that for simple moves. More precisely, we have the following

Lemma 15 *Given a three-page link diagram L and its elementary path $\gamma \subset L$, a one-sided combined move $L \rightsquigarrow L'$ described in Definition 6 such that the complexity $m(L')$ is the minimal possible can be found in $O(m(L))$ time.*

Proof. We can assume without loss of generality that γ is a connected component of $L \cap (P_1 \cup P_2)$. We describe briefly the algorithm for finding the desired one-sided move $L \rightsquigarrow L'$. The procedure consists of the following steps. First, we remove the path γ edge by edge and when deleting each internal vertex A_k of γ we look if the vertices A_{k-1} and A_{k+1} are connected by an edge in the third half-plane. If so, we simplify the diagram by elementary moves of type I and II from Proposition 2. Note that it may happen after removing the trivial edge $\overset{\frown}{A_{k-1}A_{k+1}}$ that another edge becomes trivial, as shown in Fig. 18. If so, we remove or contract it too and repeat the procedure until no trivial edges are left. Let L_1 be the obtained polygonal line.

Figure 18:

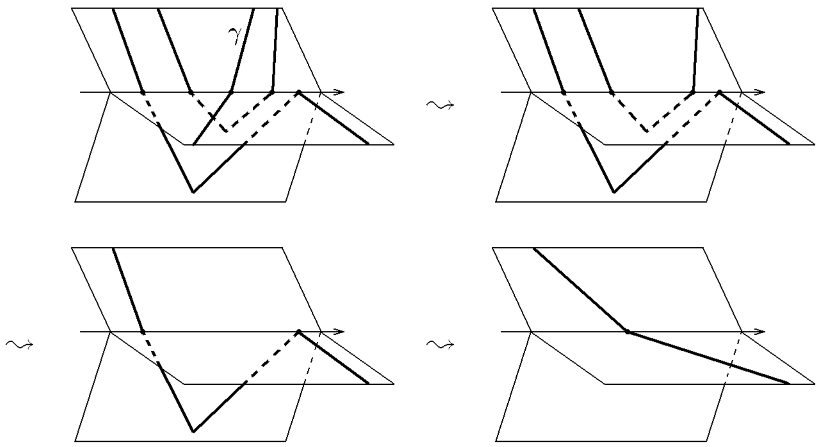
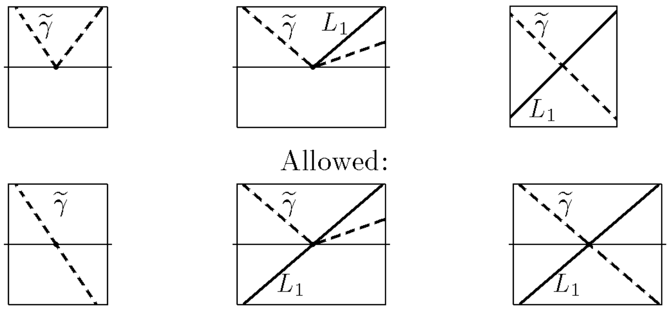


Figure 19:



Then, if the endpoints of L_1 can be connected by an elementary path lying in $P_1 \cup P_2$ and consisting of only one or two edges, we connect them. Otherwise, by analogy with finding the shortest elementary path (see the proof of Lemma 13), we find a path $\widetilde{\gamma}$ obtaining the following properties (see Fig. 19):

1) $\widetilde{\gamma} \subset P_1 \cup P_2$, $\partial\widetilde{\gamma} = \partial L_1$;

2) both sets $\widetilde{\gamma} \cap (P_1 - l)$ and $\widetilde{\gamma} \cap (P_2 - l)$ are non-empty;

3) $\widetilde{\gamma}$ is transversal to $l - (l \cap L_1)$;

4) all the intersection points of $(\widetilde{\gamma} - \partial\widetilde{\gamma})$ with L_1 lie on the line l and do not belong to $\partial(L_1 \cap (P_1 \cup P_2))$;

5) $\widetilde{\gamma} - \partial\widetilde{\gamma}$ is transversal to L_1 but may not be transversal to l at $\widetilde{\gamma} \cap L_1$;

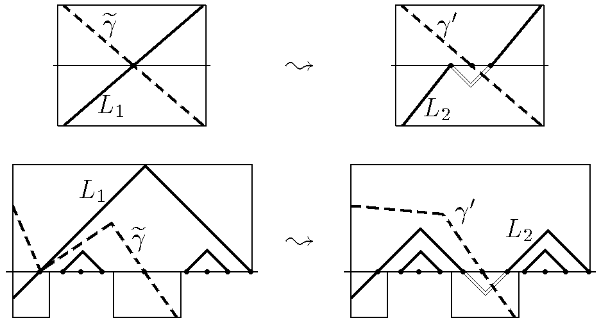
6) $\widetilde{\gamma}$ has minimum possible *length* which is defined as

$$\mathrm{length}(\widetilde{\gamma}) = \#(\text{connected components of } \widetilde{\gamma} - (l \cap \widetilde{\gamma})) + \#((\widetilde{\gamma} - \partial\widetilde{\gamma}) \cap L_1).$$

Then we replace the path $\widetilde{\gamma}$ by another path γ' isotopic to $\widetilde{\gamma}$ in $P_1 \cup P_2$ and replace L_1 by another polygonal line L_2 isotopic to L_1 in $\mathbb{Y}$, $\partial\widetilde{\gamma} = \partial\gamma' = \partial L_1 = \partial L_2$, using the rules sketched in Fig. 20 (the double line depicts edges lying in P_3). The obtained three-page diagram $L' = L_2 \cup \gamma'$ has complexity

$$\#(L_1 + \mathrm{length}(\widetilde{\gamma})).$$

Figure 20: Resolving intersection of $\widetilde{\gamma}$ with L_1.



We state that L' is (one of) the simplest three-page diagram(s) that can be obtained from L by one-sided combined move for the given γ. We leave the proof of this fact to the reader. $\square$

The notion of one-sided combined move can be naturally generalized as follows.

Definition 7. We say that a three-page link diagram L' is obtained from a three-page diagram L by *a three-sided combined move* if for *all* $i \in \mathbb{Z}_3$ there are connected components γ_i and γ_i' of $L \cap (P_{i-1} \cup P_{i+1})$ and $L' \cap (P_{i-1} \cup P_{i+1})$ respectively such that $\partial\gamma_i = \partial\gamma_i'$ and there exists an isotopy f satisfying the condition of Definition 6 if we set $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\gamma' = \gamma_1' \cup \gamma_2' \cup \gamma_3'$.

Finding a three-sided combined move $L \rightsquigarrow L'$ minimizing the complexity of L' seems to be a more complicated problem than that for one-sided moves. Untangling procedure based on three-sided combined moves has not yet been realized as a computer program and it is not clear whether three-sided combined moves are more effective for untangling than one-sided ones.

Now we shall consider another transformation which can be applied if the word w_L encoding the given three-page link diagram L has a nontrivial subword w' such that w' is balanced with respect to two of the three half-planes P_i and is not balanced with respect to the third one. Note that the existence of a completely balanced nontrivial subword means that the link is decomposable into the disjoint union of two simpler links.

Proposition 9 *Let L be a three-page link diagram such that the word w_L has the form $w_L = w_1 w_2 w_3$, where the subword w_2 is not balanced with respect to the half-plane P_i for some $i \in \mathbb{Z}_3$ and balanced with respect to the other half-planes. Then there is a simpler diagram L' representing the same link as L, $L' \sim L$, with $m(L') < m(L)$.*

PROOF. Using a cyclic permutation of the vertices $A_1, \ldots, A_{m(L)}$, we can reduce the general case to the case $w_3 = \varnothing$. Let $Q \in l$ be a point separating the binding line l into two parts $(-\infty; Q)$ and $(Q; +\infty)$ corresponding to the subwords w_1 and w_2 respectively. Let $\alpha \subset L$ be the lowest (the shortest) edge such that the left endpoint of α lies in $(-\infty; Q)$ and the right one lies in $(Q; +\infty)$.

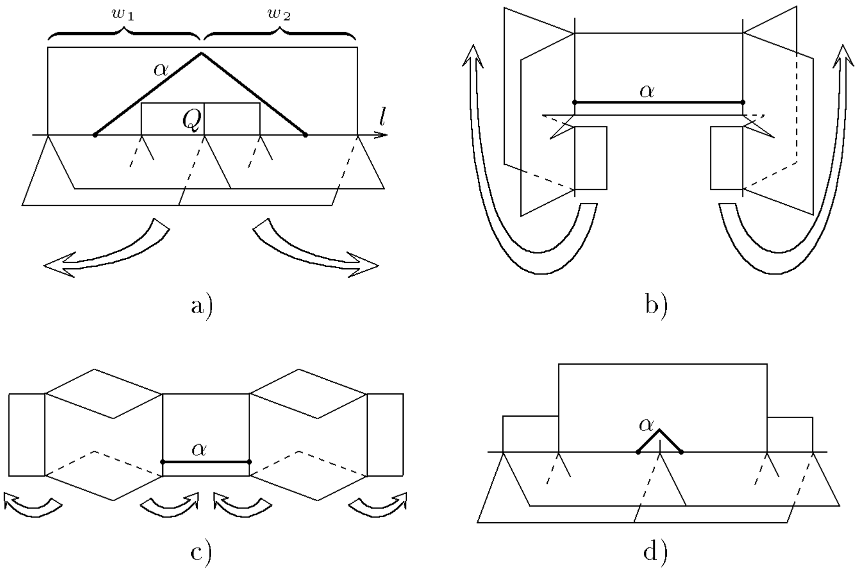
Let us cut $\mathbb{Y}$ along the lines shown in Fig. 21,a and not intersecting L. Then we deform the obtained 2-complex $\mathbb{Y}'$ together with the link L embedded in $\mathbb{Y}'$ as shown in figures 21,b–d. We indicate evolution of the edge α in the figures.

As a result of the deformation, we get a three-page link diagram L' such that its edges and vertices are in one-to-one correspondence with the edges and vertices of L. The edge of L' corresponding to the edge $\alpha \subset L$ is now trivial. This means that L' can be simplified. $\square$

We call the transformation $L \rightsquigarrow L'$ used in the proof of the proposition an *interval exchange move*.

Obviously, for given 3-page link diagram, $O(m^2)$ operations suffice for determining if the interval exchange move can be applied to the diagram, where m is its complexity. Summarizing all said above, we make the following conclusion.

Figure 21: Interval exchange move.



Theorem 6 *Given a plannar link diagram with n crossings, a 3-page link diagram L can be found for $O(n^3)$ operations such that L represents a link isotopic to the initial one and L can not be simplified by applying a one-sided combined move or an interval exchange move.*

The author of this paper has written a small and rather primitive program which implements the procedure described above. The program accepts as input (a small modification of) the Gauss code of a planar knot diagram and converts it to a three-page diagram L_0 as described in Section 7. Then it chooses an elementary path γ, removes (and forgets) it, and find the simplest three-page diagram L_1 obtained from L_0 by a one-sided combined move, using the algorithm described above. After that the program comes to the next elementary path (if we choose an orientation of the knot, the elementary paths will be naturally ordered) and perform another one-sided combined move $L_1 \rightsquigarrow L_2$, etc.

Note that the program is designed to avoid any exhaustive search which may take much time, so, in the case when it cannot get a simpler diagram in one step, it does not try find all the three-page diagrams obtained from the initial one by a finite number of one-sided moves and having the same complexity. It just tries to "shorten" each of the elementary paths.

Note also, that it may happen that $m(L_k) = m(L_{k+1})$, but $L_k \neq L_{k+1}$. The program does not control whether the next diagram coincides with the previous one. On the one hand, this makes the program simpler. On the other hand, this introduce some randomness in the simplifying process, which increases the probability of simplification of the knot.

Finally, if the program cannot find a simplifying combined move after m successive trials, where m is the complexity of the diagram, it looks if the interval exchange move can be applied to the last diagram. If so, the move is performed and simplification by one-sided combined moves is resumed. If the interval exchange move is not applicable after the complexity has stabilized, the program stops unless one of the half-planes P_i does not contain any edge. In the last case the program reports that the initial knot is trivial. The following is an example of a word representing a three-page unknot diagram (found by computer) which can be simplified neither by one one-sided combined move nor by an interval exchange move:

$$a_3a_3a_1b_2b_2a_3a_3a_2d_1d_3a_1d_3b_2d_1a_3a_3a_3d_2d_2b_3d_2d_2d_1d_1b_2d_3b_2b_2a_2b_3c_1c_1c_1b_2c_3c_1b_1c_2c_2c_1d_2d_1d_3c_2c_1c_2$$

Note, however, that the simplifying process depends essentially not only on the initial planar diagram but also on the point on the diagram from which we start writing the Gauss code. For all the examples with less than 600 crossings on which the program has been tested, it recognized the unknot for *most* of the starting points.

The unknoting program has been run for a few unknots with crossing number 1000–5000. Some of them were untangled successfully. For the others the procedure failed to find the isotopy to the circle for a few randomly chosen starting points. This does not mean that the procedure will fail for all the starting point but since each trial takes considerable time, from a few minites to two hours, it should be considered rather ineffective (from the practical point of view) for knot diagrams with crossing number > 1000.

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