Contemporary Mathematics

Cohomology of Nilmanifolds and Gontcharova's Theorem

Dmitri V. Millionschikov

Abstract. Let M_n be a nilmanifold with a corresponding nilpotent Lie algebra $\mathcal{V}_n = \{< e_1, \dots, e_n >, [e_i, e_j] = (j-i)e_{i+j}, i+j \le n\}$. We study the cohomology $H^*(M_n, \mathbb{R})$: we prove that a stable Betti number $b_q(M_n)$, $n >> q$, is equal to the $(q+2)$-th Fibonacci number F_{q+2}.

1. Introduction.

Nilmanifolds are a fruitful source of examples of symplectic manifolds with no Kähler structure or non formal symplectic manifolds (see [**TO**], [**CFG**] for references). I. Babenko and I. Taimanov described in [**BaT1**], [**BaT2**] examples of non formal simply connected symplectic manifolds. They considered an interesting family of nilmanifolds M_n that we'll discuss below. I. Babenko and I. Taimanov used only the existence of a non-trivial triple Massey product in $H^*(M_n, \mathbb{R})$ in their proof and they didn't study the cohomology $H^*(M_n, \mathbb{R})$. The question of calculation of $H^*(M_n, \mathbb{R})$ appears to be very interesting and important.

2. Nilmanifolds and symplectic structures.

Definition 2.1. A nilmanifold M is a compact homogeneous space of the form G/Γ, where G is a simply connected nilpotent Lie group and Γ is a lattice in G.

Example 2.2. A $n-$dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

Example 2.3. The Heisenberg manifold $M_3 = \mathcal{H}_3/\Gamma_3$, where $\mathcal{H}_3$ is the group of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \; x, y, z \in \mathbb{R},$$

and a lattice Γ_3 is a subgroup of matrices with $x, y, z \in \mathbb{Z}$.

Now we introduce the main example of this article.

2000 *Mathematics Subject Classification.* Primary 17B30, 17B56; Secondary 55P62, 55P20.
Key words and phrases. Nilpotent Lie algebras, cohomology, nilmanifolds.
Partially supported by the Russian Foundation for Fundamental Research, grant no. 99-01-00090.

EXAMPLE 2.4. Let $\mathcal{V}_n$ be a nilpotent Lie algebra with a basis $e_1, e_2, \dots, e_n$ and a Lie bracket:

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, i+j \le n; \\ 0, i+j > n. \end{cases}$$

One can also define a group structure $*$ in the vector space $\mathcal{V}_n$ using the Campbell-Hausdorff formula:

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] - [y,[y,x]]) + \dots.$$

This nilpotent group $V_n = (\mathcal{V}_n, *)$ has a discrete subgroup (lattice) Γ_n and according to A.Maltsev's theorem ([**Mal**]) quotient spaces $M_n = V_n/\Gamma_n$ are compact manifolds (nilmanifolds).

REMARK 2.5. The manifolds M_n are Eilenberg-MacLane spaces $K(\Gamma_n, 1)$.

It was Victor Buchstaber who introduced these manifolds in [**Bu1**], [**Bu2**] for the study of the Landweber-Novikov algebra. V. Buchstaber proposed also a very interesting interpretation of V_n as a group of polynomial real line infinitesimal transformations

$$\{t \to P_{\mathbf{x}}(t) = t + x_1 t^2 + \dots + x_n t^{n+1}\}$$

with a nilpotent group structure defined by

$$(P_{\mathbf{x}} * Q_{\mathbf{y}})(t) := Q_{\mathbf{y}}(P_{\mathbf{x}}(t)) \bmod t^{n+2},$$

where we denote by $\mathbf{x} = (x_1, x_2, \dots, x_n)$ a vector in the $\mathcal{V}_n$.

The cohomology $H^*(M_n, \mathbb{R})$ is isomorphic to the cohomology $H^*(\mathcal{V}_n)$ of the corresponding Lie algebra $\mathcal{V}_n$ (Nomizu theorem [**Nz**]). One of the most important Babenko-Taimanov's observations was

REMARK 2.6. ([**BaT1**]) M_{2k} is a symplectic manifold with a left-invariant symplectic form

$$\Omega_{2k+1} = (2k-1)\omega_1 \wedge \omega_{2k} + (2k-3)\omega_2 \wedge \omega_{2k-1} + \dots + \omega_k \wedge \omega_{k+1},$$

where $\omega_1, \dots, \omega_{2k}$ is a dual basis of left-invariant 1-forms to the $e_1, \dots, e_{2k}$.

REMARK 2.7. A rational minimal model $\mathcal{M}_{M_n}$ for M_n has generators $x_1, \dots, x_n$ of degree 1, with a differential

$$dx_1 = 0, dx_2 = 0, dx_k = \sum_{i<j, i+j=k} (j-i) x_i \wedge x_j, k = 3, \dots n.$$

In fact I. Babenko, I. Taimanov in [**BaT1**], [**BaT2**] applied the symplectic blow-up procedure to the symplectic embedding $M_{2k} \to \mathbb{C}P^{2k+1}$ (as it was done by D. McDuff in [**McD**] where she described the first example of simply connected symplectic non Kähler manifold).

3. Cohomology of infinite dimensional Lie algebras and Gontcharova's theorem.

The problem of calculation of the cohomology of the $\mathcal{V}_n$ is closely related to Gontcharova's theorem in the infinite dimensional Lie algebras theory.

DEFINITION 3.1. L_k is an infinite dimensional Lie algebra of polynomial vector fields on $\mathbb{R}$ with a trivial k-jet at the origin. L_k has a basis

$$e_i = x^{i+1}\frac{d}{dx}, i = k, k+1, \dots; \; [e_i, e_j] = (j-i)e_{i+j}.$$

So the algebra $\mathcal{V}_n$ is a quotient algebra $\mathcal{V}_n = L_1/L_{n+1}$.

The problem of calculation of $H^*(L_k)$ was solved by Lidiya Gontcharova in 1973. In [**G**] Gontcharova used the second grading p of a complex of continuous cochains $C^*(L_k)$ $(p(\omega_{i_1}\wedge\ldots\wedge\omega_{i_q}) = i_1+\cdots+i_q)$ for her computation of the bigraded cohomology $H^q_p(L_k)$.

THEOREM 3.2. *(L.Gontcharova,[**G**])*

$$1)\ \dim H^q_k(L_1) = \begin{cases} 1, & \text{if } k = \frac{3q^2\pm q}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\dim H^q(L_1) = 2$ *for all* $q \geq 1$.

$$2)\ \dim H^q(L_k) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-1}.$$

REMARK 3.3. One can consider a formula for Euler characteristic χ_N of the complex $C^*_N(L_1)$ of cochains of the second grading $p = N$ and then define a corresponding generating function:

$$\sum_N \chi_N t^N = \prod_{n=1}^{\infty}(1-t^n) = 1 - t - t^2 + t^5 + t^7 + \ldots + (-1)^q(t^{\frac{3q^2-q}{2}} + t^{\frac{3q^2+q}{2}}) + \ldots$$

This is the well-known Euler identity in combinatorics. From topological point of view this identity is an equality of two expressions of Euler characteristic χ_N (the right part of the formula is an alternated sum of Betti numbers, see [**Fu**] for details).

Numbers $\frac{3q^2\pm q}{2}$ are so called Euler pentagonal numbers. From their properties it follows that the algebra $H^*(L_1)$ has a trivial multiplication (see [**FFR**]).

THEOREM 3.4. *A stable Betti number* $b_q(M_n) = \dim(H^q(\mathcal{V}_n))$, $n >> q$, *is equal to the* $(q+2)$*-th Fibonacci number* F_{q+2} :

$$b_1(M_n) = \dim(H^1(\mathcal{V}_n)) = 2,\ b_2(M_n) = \dim(H^2(\mathcal{V}_n)) = 3,$$

$$b_3(M_n) = \dim(H^3(\mathcal{V}_n)) = 5, \ldots, b_q(M_n) = \dim(H^q(\mathcal{V}_n)) = F_{q+2}, n >> q.$$

PROOF. (Sketch of the proof). Let us consider the Serre-Hochschild spectral sequence of a pair of Lie algebras (L_1, L_{n+1}) (see [**HS**]) that converges to the $H^*(L_1)$.

In fact the first row of E_2 is our answer. We compute here $H^q(L_1/L_{n+1})$ only for $n >> q$.

LEMMA 3.5. $H^q(L_1) = E^{q,0}_\infty = E^{q,0}_3$. *In other words: Gontcharova's cocycles of small degrees* $q << n$ *are in the first row of* E_2 *and survive till* E_∞.

LEMMA 3.6. $H^q(L_1/L_{n+1}) = Im d_2 H^{q-2}(L_1/L_{n+1}, H^1(L_{n+1})) \oplus H^q(L_1)$. *In other words: to get a basis of* $H^q(L_1/L_{n+1})$ *for large values* $n >> q$ *we have to add to two Gontcharova's cocycles of degree* q *a basis of* $Im d_2 H^{q-2}(L_1/L_{n+1}, H^1(L_{n+1}))$.

EXAMPLE 3.7. Let us consider dimensions $q = 1, 2, 3$ and $n >> q$:

1) $H^1(\mathcal{V}_n)$ is generated by two classes $[\omega_1]$ and $[\omega_2]$ – Gontcharova's classes $g_1 = [\omega_1], g_2 = [\omega_2]$.

$$E_3 = E_4 = \cdots = E_\infty.$$

0	0	0	0	...
0	0	0	0	...
0	ω_1, ω_2	g_5, g_7	g_{12}, g_{15}	...

FIGURE 1. Gontcharova's cocycles and E_∞.

$\omega_{n+1} \wedge \omega_{n+4}$	d_2		d_2	
ω_{n+1}	$\omega_2 \wedge \omega_{n+1}$ $\omega_2 \wedge \omega_{n+2} + ..$ $\omega_2 \wedge \omega_{n+3} + ..$	$g_5 \wedge \omega_{2n} + ..$ $g_7 \wedge \omega_{n+1}$ $\cdots$ $g_7 \wedge \omega_{n+5} + ..$ $\Omega_{n+1} \wedge \omega_{n+1}$		
0	ω_1, ω_2	Ω_{n+1} g_5, g_7	$\omega_2 \wedge \Omega_{n+1}$ $\omega_2 \wedge \Omega_{n+2} + ..$ $\omega_2 \wedge \Omega_{n+3} + ..$ g_{12}, g_{15}	$g_7 \wedge \Omega_{n+1}$ $\cdots$ $g_7 \wedge \Omega_{n+5} + ..$ $\Omega_{n+1}^2, g_{22}, g_{26}$

FIGURE 2. The term E_2 of the spectral sequence.

2) The basis of $H^2(\mathcal{V}_n)$ consists of two Gontcharova's classes $g_5 = [\omega_1 \wedge \omega_4]$, $g_7 = [\omega_2 \wedge \omega_5 + \frac{3}{2} \omega_1 \wedge \omega_6]$ and the third generator is the "symplectic class" $[\Omega_{n+1}]$, where $\Omega_{n+1} = (n-1)\omega_1 \wedge \omega_n + (n-3)\omega_2 \wedge \omega_{n-1} + \ldots$.

3) In $H^3(\mathcal{V}_n)$ to the classes $g_{12} = [\omega_1 \wedge \omega_4 \wedge \omega_7 + \ldots], g_{15} = [\omega_2 \wedge \omega_5 \wedge \omega_8 + \ldots]$ (degree 3 with second gradings p=12,15 respectively, see [**G**] for details) we add three new ones

$$[\omega_2] \wedge [\Omega_{n+1}], [\omega_2 \wedge \Omega_{n+2} - n\omega_3 \wedge \Omega_{n+1}],$$

$$[\omega_2 \wedge \Omega_{n+3} - (n+1)\omega_3 \wedge \Omega_{n+2} + \frac{n(n+1)}{2} \omega_4 \wedge \Omega_{n+1}],$$

where $\Omega_{n+2}, \Omega_{n+3}$ are projections of "non-existing" $d\omega_{n+2}, d\omega_{n+3}$ to the $C^2(\mathcal{V}_n)$:

$$\Omega_{n+2} = (n-2)\omega_2 \wedge \omega_n + (n-4)\omega_3 \wedge \omega_{n-1} + \ldots; \Omega_{n+3} = (n-3)\omega_3 \wedge \omega_n + \ldots.$$

REMARK 3.8. An elementary calculation shows that the cohomology algebra $H^*(\mathcal{V}_n)$ does not satisfy the hard Lefschetz condition:

$$[\omega_1] \wedge [\Omega_{n+1}] = 0, \ \omega_1 \wedge \Omega_{n+1} = \frac{1}{n} d\Omega_{n+2}.$$

LEMMA 3.9.

$$H^q(\mathcal{V}_n) = H^q(L_1) \oplus Imd_2 = H^q(L_1) \oplus H^{q-2}(L_2) \oplus H^{q-4}(L_3) \oplus \ldots.$$

The last proposition together with Gontcharova's theorem concludes the proof, namely, it gives us a representation of Fibonacci numbers by means of binomial coefficients:

$$F_{q+2} = \binom{q}{0} + \binom{q-1}{0} + \binom{q-1}{1} + \binom{q-2}{1} + \binom{q-2}{2} + \binom{q-3}{2} + \dots .$$

It is a finite sum that can be illustrated by means of the Pascal triangle:

$$\begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} * & & & & \\ * & * & & & \\ * & 2 & 1 & & \\ 1 & 3 & * & * & \\ 1 & * & * & * & * \end{array} .$$

EXAMPLE 3.10. Let us consider a sum of the fourth and the third diagonals of the Pascal triangle

$$F_6 = \{1+3+1\} + \{1+2\} = 8,$$

or in terms of binomial coefficients

$$F_6 = F_{4+2} = \left\{ \binom{4}{0} + \binom{3}{1} + \binom{2}{2} \right\} + \left\{ \binom{3}{0} + \binom{2}{1} \right\} = 8.$$

□

References

[BaT1] I.K. Babenko and I.A. Taimanov. *On the existence of nonformal simply connected symplectic manifolds*, Russian Math. Surveys **53**:4 (1998), 1082-1083.

[BaT2] I.K. Babenko and I.A. Taimanov. *On nonformal simply connected symplectic manifolds*, SFB 2888, Preprint 358, 1998.

[Bu1] V.M. Buchstaber, A.V. Shokurov, *The Landweber–Novikov algebra and formal vector fields on the line*, Funct. Anal. and Appl. **12**:3 (1978), 1–11.

[Bu2] V.M. Buchstaber, *The groups of polynomial transformations of the line, nonformal symplectic manifolds, and the Landweber-Novikov algebra*, Russian Math. Surveys **54** (1999).

[CFG] L.A. Cordero, M. Fernandez, A. Gray, *Symplectic manifolds with no Kähler structure*, Topology **25** (1986), 375–380.

[DGMS] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler structure*, Invent. Math. **19** (1975), 245–274.

[FFR] B.L. Feigin, D.B. Fuchs, V.S. Retakh, *Massey operations in the cohomology of the infinite dimensional Lie algebra L_1*, Rokhlin Seminar, Lecture Notes in Math. **1346** (1988), 13–31.

[Fu] D.B. Fuchs, *Cohomology of the infinitedimensional Lie algebras*, Consultant Bureau, New York, 1987.

[G] L.V. Gontcharova, *Cohomology of Lie algebras of formal vector fields on the line*, Funct. Anal. and Appl. **7**:2 (1973), 6–14.

[HS] G.P. Hochschild, J.-P. Serre, *Cohomology of Lie algebras*, Ann. Math. **57**:3 (1953), 367–401.

[Mal] A. Malcev, *On a class of homogeneous spaces*, Izvestia Akad. Nauk SSSR Ser. Mat. **3** (1949), 9-32 (Russian); English translation: Amer. Math. Soc. Transl. (1) **9** (1962), 276–307.

[McD] D. McDuff, *Examples of symplectic simply-connected manifolds with no Kähler structure*, J. Diff. Geom. **20** (1984), 267–277.

[Mill] D.V. Millionschikov, *Cohomology of nilmanifolds and Gontcharova's theorem*, Russian Math. Surveys, to appear.

[Nz] K. Nomizu, *On the cohomology of homogeneous spaces of nilpotent Lie groups*, Ann. of Math. **59** (1954), 531–538.

[TO] A. Tralle, J. Oprea, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Math. **1661** (1997).

DEPARTMENT OF MATHEMATICS AND MECHANICS, MOSCOW STATE UNIVERSITY, 119899 MOSCOW, RUSSIA

E-mail address: `million@mech.math.msu.su`