TORUS ACTIONS, EQUIVARIANT MOMENT-ANGLE COMPLEXES, AND COORDINATE SUBSPACE ARRANGEMENTS

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Abstract. We show that the cohomology algebra of the complement of a coordinate subspace arrangement in m-dimensional complex space is isomorphic to the cohomology algebra of Stanley–Reisner face ring of a certain simplicial complex on m vertices. (The face ring is regarded as a module over the polynomial ring on m generators.) Then we calculate the latter cohomology algebra by means of the standard Koszul resolution of polynomial ring. To prove these facts we construct an equivariant with respect to the torus action homotopy equivalence between the complement of a coordinate subspace arrangement and the moment-angle complex defined by the simplicial complex. The moment-angle complex is a certain subset of a unit poly-disk in m-dimensional complex space invariant with respect to the action of an m-dimensional torus. This complex is a smooth manifold provided that the simplicial complex is a simplicial sphere, but otherwise has more complicated structure. Then we investigate the equivariant topology of the moment-angle complex and apply the Eilenberg–Moore spectral sequence. We also relate our results with well known facts in the theory of toric varieties and symplectic geometry.

1. Introduction

In this paper we apply the results of our previous paper [BP2] to describing the topology of the complement of a complex coordinate subspace arrangement. A coordinate subspace arrangement $\mathcal{A}$ is a set of coordinate subspaces L of a complex space $\mathbb{C}^m$, and its complement is the set $U(\mathcal{A}) = \mathbb{C}^m \setminus \bigcup_{L\in\mathcal{A}} L$. The complement $U(\mathcal{A})$ decomposes as $U(\mathcal{A}) = U(\mathcal{A}') \times (\mathbb{C}^*)^k$, were $\mathcal{A}'$ is a coordinate arrangement in $\mathbb{C}^{m-k}$ that does not contain any hyperplane. There is a one-to-one correspondence between coordinate subspace arrangements in $\mathbb{C}^m$ without hyperplanes and simplicial complexes on m vertices $v_1, \dots, v_m$: each arrangement $\mathcal{A}$ defines a simplicial complex $K(\mathcal{A})$ and vice versa. Namely let $|\mathcal{A}|$ denotes the support $\bigcup_{L\in\mathcal{A}} L$ of the coordinate subspace arrangement $\mathcal{A}$; then a subset $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is a $(k-1)$-simplex of $K(\mathcal{A})$ if and only if the $(m-k)$-dimensional coordinate subspace $L_I \subset \mathbb{C}^m$ defined by equations $z_{i_1} = \dots = z_{i_k} = 0$ does not belong

1991 *Mathematics Subject Classification.* 55N91, 05B35 (Primary) 13D03 (Secondary).

Partially supported by the Russian Foundation for Fundamental Research, grant no. 99-01-00090, and INTAS, grant no. 96-0770.

to $|\mathcal{A}|$. An arrangement $\mathcal{A}$ is obviously recovered from its simplicial complex $K(\mathcal{A})$; that is why we write $U(K)$ instead of $U(\mathcal{A}(K))$ throughout this paper. (For more information about relations between arrangements and simplicial complexes see the beginning of Section 2.)

Subspace arrangements and their complements play a pivotal role in many constructions of combinatorics, algebraic and symplectic geometry, mechanics etc., they also arise as configuration spaces of different classical systems. That is why the topology of complements of arrangements entranced many mathematicians during the last two decades. The first important result here deals with arrangements of hyperplanes (not necessarily coordinate) in $\mathbb{C}^m$. Arnold [Ar] and Brieskorn [Br] shown that the cohomology algebra of the corresponding complement $U(\mathcal{A})$ is isomorphic to the algebra of differential forms generated by the closed forms $\frac{1}{2\pi i}\frac{dF_A}{F_A}$, where F_A is a linear form defining the hyperplane A of the arrangement. Orlik and Solomon [OS] proved that the cohomology algebra of the complement of a hyperplane arrangement depends only on the combinatorics of intersections of hyperplanes and presented $H^*(U(\mathcal{A}))$ by generators and relations. In general situation, the Goresky–MacPherson theorem [GM, Part III] expresses the cohomology groups $H^i(U(\mathcal{A}))$ (without ring structure) as a sum of homology groups of subcomplexes of a certain simplicial complex. This complex, called the *order* (or *flag*) complex, is defined via the combinatorics of intersections of subspaces of $\mathcal{A}$. The proof of this result uses the stratified Morse theory developed in [GM]. Another way to handle the cohomology algebra of the complement of a subspace arrangement was recently presented by De Concini and Procesi [dCP]. They proved that the rational cohomology ring of $U(\mathcal{A})$ is also determined by the combinatorics of intersections. This result was extended by Yuzvinsky in [Yu]. In the case of coordinate subspace arrangements the order complex is the barycentric subdivision of a simplicial complex $\tilde{K}$, while the summands in the Goresky–MacPherson formula are homology groups of links of simplices of $\tilde{K}$. The complex $\tilde{K}$ has the same vertex set $v_1, \dots, v_m$ as our simplicial complex K and is "dual" to the latter in the following sense: a set $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ spans a simplex of $\tilde{K}$ if and only if the complement $\{v_1, \dots, v_m\} \setminus v_I$ does not span a simplex of K. The product of cohomology classes of the complement of a coordinate subspace arrangement was described in [dL] in combinatorial terms using the complex $\tilde{K}$ and the above interpretation of the Goresky–MacPherson formula.

In our paper we prefer to describe a coordinate subspace arrangement in terms of the simplicial complex K instead of $\tilde{K}$ because such an approach reveals new connections between the topology of complements of subspace arrangements, commutative algebra, and geometry of *toric varieties*. We show that the complement $U(K)$ is homotopically equivalent to what we call the *moment-angle complex* $\mathcal{Z}_K$ defined by the simplicial complex K. This $\mathcal{Z}_K$ is a compact subset of a unit poly-disk $(D^2)^m \subset \mathbb{C}^m$ invariant with respect to the standard T^m-action on $(D^2)^m$. At the same time $\mathcal{Z}_K$ is a homotopy fibre of cellular embedding $i : \widetilde{B_T K} \hookrightarrow BT^m$, where BT^m is the T^m-classifying space with standard cellular structure, and $\widetilde{B_T K}$ is a cell subcomplex whose cohomology is isomorphic to the *Stanley–Reisner face ring* $\mathbf{k}(K)$ of simplicial complex K. Then we calculate the cohomology algebra of $\mathcal{Z}_K$ (or $U(K)$) by means of the Eilenberg–Moore spectral sequence. As the result, we obtain an algebraic description of the cohomology algebra of $U(K)$ as the bigraded cohomology algebra $\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ of the face ring $\mathbf{k}(K)$. By means of the standard Koszul resolution the latter can be expressed as the cohomology of differential bigraded algebra $\mathbf{k}(K)\otimes\Lambda[u_1,\dots,u_m]$, where $\Lambda[u_1,\dots,u_m]$ is an exterior algebra, and the differential sends exterior generator u_i to $v_i \in \mathbf{k}(K) = \mathbf{k}[v_1,\dots,v_m]/I$. The rational models of De Concini and Procesi [dCP] and Yuzvinsky [Yu] also can be interpreted as an application of the Koszul resolution to the cohomology of the complement a subspace arrangement, however the role of the face ring became clear only after our paper [BP2].

If K is an $(n-1)$-dimensional simplicial sphere (for instance, K is the boundary complex of an n-dimensional convex simplicial polytope), our moment-angle complex $\mathcal{Z}_K$ turns to be a smooth $(m+n)$-dimensional manifold (hence, $U(K)$ is homotopically equivalent to a smooth manifold). This important particular case of our constructions was detailedly studied in [BP1], [BP2]. Topological properties of the above manifolds $\mathcal{Z}_K$ are of great interest because of their relations with combinatorics of polytopes, symplectic geometry, and geometry of toric varieties; the last thing was the starting point in our study of coordinate subspace arrangements. The classical definition of toric varieties (see [Da], [Fu]) deals with the combinatorial object known as *fan*. However, as it have been recently shown by several authors (see, for example, [Au], [Ba], [Co]), in the case when the fan defining a toric variety M is simplicial, M can be defined as the geometric quotient of the complement $U(K)$ with respect to a certain action of the algebraic torus $(\mathbb{C}^*)^{m-n}$ (here K is the simplicial complex defined by the fan). Our moment-angle manifold $\mathcal{Z}_K$ is the pre-image of a regular point in the image of the moment map $U(K)\to\mathbb{R}^{m-n}$ for the Hamiltonian action of compact torus $T^{m-n}\subset(\mathbb{C}^*)^{m-n}$.

In their paper [DJ] Davis and Januszkiewicz introduced the notion of toric manifold (now also known as quasitoric manifold or unitary toric manifold), which can be regarded as a natural topological extension of the notion of smooth toric variety. A (quasi)toric manifold M^{2n} admits a smooth action of the torus T^n that locally looks like the standard action of T^n on $\mathbb{C}^n$; the orbit space is required to be an n-dimensional ball, invested with the combinatorial structure of a simple convex polytope by the fixed point sets of appropriate subtori. Topology, geometry and combinatorics of quasitoric manifolds are very beautiful; after the pioneering paper [DJ] many new relations have been discovered by different authors (see [BR1], [BR2], [BP1], [BP2], [Pa1], [Pa2], and more references there). The dual complex to the boundary complex of a simple polytope in the orbit space of a quasitoric manifold is a simplicial sphere. That is why many results from the present paper may be considered as an extension of our previous constructions with simplicial spheres to the case of general simplicial complex. We also mention that some our definitions and constructions (such as the Borel construction $B_T P$) firstly appeared in [DJ] in a different fashion; in this case we have tried to retain initial notations.

The authors express special thanks to Nigel Ray for stimulating discussions and fruitful collaboration which inspired some ideas and constructions from this paper. We also grateful to Nataliya Dobrinskaya who have drawn our attention to paper [Ba], which reveals some connections between toric varieties and coordinate subspace arrangements, and to Sergey Yuzvinsky who informed us about the results of preprint [dL].

2. Homotopical realization of complement of a coordinate subspace arrangement

Let $\mathbb{C}^m$ be a complex m-dimensional space with coordinates $z_1, \dots, z_m$. For any index subset $I = \{i_1, \dots, i_k\}$ denote by L_I the $(m-k)$-dimensional coordinate subspace defined by the equations $z_{i_1} = \dots = z_{i_k} = 0$. Note that $L_{\{1,\dots,m\}} = \{0\}$ and $L_\varnothing = \mathbb{C}^m$.

Definition 2.1. A *coordinate subspace arrangement* $\mathcal{A}$ is a set of coordinate subspaces L_I. The *complement* of $\mathcal{A}$ is the subset

$$U(\mathcal{A}) = \mathbb{C}^m \setminus \bigcup_{L_I \in \mathcal{A}} L_I \subset \mathbb{C}^m.$$

In the sequel we would distinguish the coordinate subspace arrangement $\mathcal{A}$ regarded as an abstract set of subspaces and its support $|\mathcal{A}|$ — the subset $\bigcup_{L_I \in \mathcal{A}} L_I \subset \mathbb{C}^m$. If $I \subset J$ and $L_I \subset |\mathcal{A}|$, then $L_J \subset |\mathcal{A}|$. If a coordinate subspace arrangement $\mathcal{A}$ contains a hyperplane $z_i = 0$, then its complement $U(\mathcal{A})$ is represented as $U(\mathcal{A}_0) \times \mathbb{C}^*$, where $\mathcal{A}_0$ is a coordinate subspace arrangement in the hyperplane $\{z_i = 0\}$, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus, for any coordinate subspace arrangement $\mathcal{A}$ the complement $U(\mathcal{A})$ decomposes as

$$U(\mathcal{A}) = U(\mathcal{A}') \times (\mathbb{C}^*)^k,$$

were $\mathcal{A}'$ is a coordinate arrangement in $\mathbb{C}^{m-k}$ that does not contain any hyperplane. Keeping in mind this remark, we restrict ourself to coordinate subspace arrangement without hyperplanes.

A coordinate subspace arrangement $\mathcal{A}$ in $\mathbb{C}^m$ (without hyperplanes) defines a simplicial complex $K(\mathcal{A})$ with m vertices $v_1, \dots, v_m$ in the following way: we say that a subset $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is a $(k-1)$-simplex of $K(\mathcal{A})$ if and only if $L_I \not\subset |\mathcal{A}|$.

Example 2.2. 1) If $\mathcal{A} = \varnothing$, then $K(\mathcal{A})$ is an $(m-1)$-dimensional simplex Δ^{m-1}.

2) If $\mathcal{A} = \{0\}$, then $K(\mathcal{A}) = \partial\Delta^{m-1}$ is the boundary of an $(m-1)$-simplex.

On the other hand, a simplicial complex K on the vertex set $\{v_1,\ldots,v_m\}$ defines an arrangement $\mathcal{A}(K)$ such that $L_I\subset|\mathcal{A}|$ if and only if $v_I=\{v_{i_1},\ldots,v_{i_k}\}$ is not a simplex of K. Note that if $K'\subset K$ is a subcomplex, then $\mathcal{A}(K)\subset\mathcal{A}(K')$. Thus, we have a reversing order one-to-one correspondence between simplicial complexes on m vertices and coordinate subspace arrangements in $\mathbb{C}^m$ without hyperplanes.

Now let $U(K)=\mathbb{C}^m\setminus|\mathcal{A}(K)|$ denote the complement of the coordinate subspace arrangement $\mathcal{A}(K)$.

Example 2.3. 1) If $K=\Delta^{m-1}$ is an $(m-1)$-simplex, then $U(K)=\mathbb{C}^m$.

2) If $K=\partial\Delta^{m-1}$, then $U(K)=\mathbb{C}^m\setminus\{0\}$.

3) If K is a disjoint union of m vertices, then $U(K)$ is obtained by removing from $\mathbb{C}^m$ all codimension-two coordinate subspaces $z_i=z_j=0$, $i,j=1,\ldots,m$.

Suppose that $\mathbf{k}$ is any field, which we refer to as the ground field. Form a polynomial ring $\mathbf{k}[v_1,\ldots,v_m]$ where the v_i are regarded as indeterminates.

Definition 2.4. The *face ring* (or the *Stanley–Reisner ring*) $\mathbf{k}(K)$ of simplicial complex K is the quotient ring $\mathbf{k}[v_1,\ldots,v_m]/I$, where

$$I=(v_{i_1}\cdots v_{i_s}:\{v_{i_1},\ldots,v_{i_s}\}\text{ does not span a simplex in }K).$$

Thus, the face ring is a quotient ring of polynomial ring by an ideal generated by square free monomials of degree ≥ 2. We make $\mathbf{k}(K)$ a graded ring by setting $\deg v_i=2$, $i=1,\ldots,m$.

Example 2.5. 1) If $K=\Delta^{m-1}$, then $\mathbf{k}(K)=\mathbf{k}[v_1,\ldots,v_m]$.

2) If $K=\partial\Delta^{m-1}$ is the boundary complex of a $(m-1)$-simplex, then $\mathbf{k}(K)=\mathbf{k}[v_1,\ldots,v_m]/(v_1\cdots v_m)$.

A compact torus T^m acts on $\mathbb{C}^m$ diagonally; since the arrangement $\mathcal{A}(K)$ consists of coordinate subspaces, this action is also defined on $U(K)$. Denote by B_TK the corresponding Borel construction:

$$B_TK=ET^m\times_{T^m}U(K), \tag{1}$$

where ET^m is the contractible space of universal T^m-bundle $ET^m\to BT^m$ over the classifying space $BT^m=(\mathbb{C}P^\infty)^m$. Thus, B_TK is the total space of bundle $B_TK\to BT^m$ with fibre $U(K)$.

The space BT^m has a canonical cellular decomposition (that is, each $\mathbb{C}P^\infty$ has one cell in each even dimension). For each index set $I=\{i_1,\ldots,i_k\}$ one may consider the cellular subcomplex $BT^k_I=BT^k_{i_1,\ldots,i_k}\subset BT^m$ homeomorphic to BT^k.

Definition 2.6. Given a simplicial complex K with vertex set $\{v_1,\ldots,v_m\}$, define cellular subcomplex $\widetilde{B_TK}\subset BT^m$ as the union of BT^k_I over all I such that v_I is a simplex of K.

Example 2.7. Let K be a disjoint union of m vertices $v_1,\ldots,v_m$. Then $\widetilde{B_TK}$ is a bouquet of m copies of $\mathbb{C}P^\infty$.

The cohomology ring of BT^m is isomorphic to the polynomial ring $\mathbf{k}[v_1,\dots,v_m]$ (all cohomologies are with coefficients in the ground field $\mathbf{k}$).

Lemma 2.8. *The cohomology ring of $\widetilde{B_TK}$ is isomorphic to the face ring $\mathbf{k}(K)$. The embedding $i:\widetilde{B_TK}\hookrightarrow BT^m$ induces the quotient epimorphism $i^*:\mathbf{k}[v_1,\dots,v_m]\to\mathbf{k}(K)=\mathbf{k}[v_1,\dots,v_m]/I$ in the cohomology.*

Proof. The proof is by induction on the number of simplices of K. If K is a disjoint union of vertices $v_1,\dots,v_m$, then $\widetilde{B_TK}$ is a bouquet of m copies of $\mathbb{C}P^\infty$ (see Example 2.7). In degree zero $H^*(\widetilde{B_TK})$ is just $\mathbf{k}$, while in degrees ≥ 1 it is isomorphic to $\mathbf{k}[v_1]\oplus\dots\oplus\mathbf{k}[v_m]$. Therefore, $H^*(\widetilde{B_TK})=k[v_1,\dots,v_m]/I$, where I is the ideal generated by all square free monomials of degree ≥ 2, and i^* is the projection onto the quotient ring. Thus, the lemma holds for $\dim K=0$.

Now suppose that the simplicial complex K is obtained from the simplicial complex K' by adding one $(k-1)$-dimensional simplex $v_I=\{v_{i_1},\dots,v_{i_k}\}$. By the inductive hypothesis, the lemma holds for K', that is, $i^*H^*(BT^m)=H^*(\widetilde{B_TK'})=\mathbf{k}(K')=\mathbf{k}[v_1,\dots,v_m]/I'$. By Definition 2.6, $\widetilde{B_TK}$ is obtained from $\widetilde{B_TK'}$ by adding the subcomplex $BT^k_{i_1,\dots,i_k}\subset BT^m$. Then $H^*(\widetilde{B_TK'}\cup BT^k_{i_1,\dots,i_k})=\mathbf{k}[v_1,\dots,v_m]/I=\mathbf{k}(K'\cup v_I)$, where I is generated by I' and $v_{i_1}v_{i_2}\cdots v_{i_k}$. □

Let I^m be the standard m-dimensional cube in $\mathbb{R}^m$:

$$I^m=\{(y_1,\dots,y_m)\in\mathbb{R}^m\ :\ 0\leq y_i\leq 1,\ i=1,\dots,m\}.$$

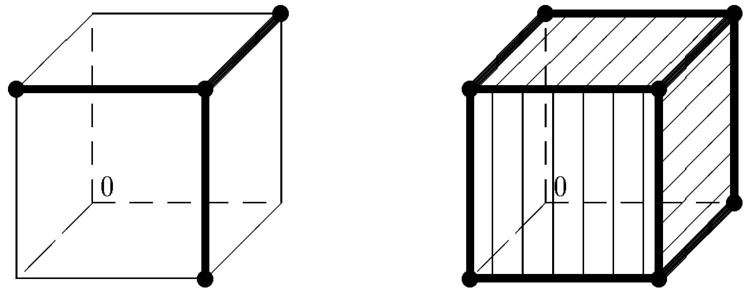
A simplicial complex K with m vertices $v_1,\dots,v_m$ defines a cubical complex $\mathcal{C}_K$ embedded canonically into the boundary complex of I^m in the following way:

Definition 2.9. For each $(k-1)$-dimensional simplex $v_J=\{v_{j_1},\dots,v_{j_k}\}$ of K denote by C_J the k-dimensional face of I^m defined by $m-k$ equations

$$y_i=1,\quad i\notin\{j_1,\dots,j_k\}.$$

Then define cubical subcomplex $\mathcal{C}_K\subset I^m$ as the union of C_J over all simplices v_J of K.

Remark. Our cubical subcomplex $\mathcal{C}_K\subset I^m$ is a geometrical realization of an abstract cubical complex in the cone over the barycentric subdivision of K (see [DJ, p. 434]). Indeed, let Δ^{m-1} be an $(m-1)$-dimensional simplex on the vertex set $\{v_1,\dots,v_m\}$, and $\hat\Delta^{m-1}$ a barycentric subdivision of Δ^{m-1}, that is, $\hat\Delta^{m-1}$ has a vertex for each simplex v_J of Δ^{m-1}. Construct a map $\iota:\hat\Delta^{m-1}\to I^m$ by sending vertex v_J of $\hat\Delta^{m-1}$ to the vertex of I^m having coordinates $y_j=0$ for $j\in J$ and $y_j=1$ for $j\notin J$ and then extending this map linearly on each simplex of $\hat\Delta^{m-1}$. The image of $\hat\Delta^{m-1}$ under the constructed map is the union of faces of I^m meeting at zero. Then build a map $C\iota$ from the cone $C\hat\Delta^{m-1}$ over $\hat\Delta^{m-1}$ to I^m by sending the vertex of the cone to the vertex $(1,\dots,1)$ of the cube and extending linearly on simplices



a) b)

FIGURE 1. The cubical complex $\mathcal{C}_K$.

of $C\hat{\Delta}^{m-1}$. The image of $C\hat{\Delta}^{m-1}$ under the map $C\iota$ is the whole cube I^m. Now let K be a simplicial complex on the vertex set $\{v_1, \dots, v_m\}$. Once a numeration of vertices is fixed, we may view K as a simplicial subcomplex of Δ^{m-1}. Then our cubical complex $\mathcal{C}_K \subset I^m$ from Definition 2.9 is nothing but the image $C\iota(C\hat{K})$ of the cone over the barycentric subdivision of K under the map $C\iota$.

Example 2.10. The cubical complex $\mathcal{C}_K$ in the cases when K is a disjoint union of 3 vertices and the boundary complex of a 2-simplex is indicated on Figure 1 a) and b) respectively.

Remark. In the case when K is the dual to the boundary complex of an n-dimensional simple polytope P^n, the cubical complex $\mathcal{C}_K$ coincides with the cubical subdivision of P^n studied in [BP2].

The orbit space of the diagonal action of T^m on $\mathbb{C}^m$ is the positive cone

$$\mathbb{R}^m_+ = \{(y_1, \dots, y_m) \in \mathbb{R}^m \,:\, y_i \geq 0,\, i = 1, \dots, m\}.$$

The orbit map $\mathbb{C}^m \to \mathbb{R}^m_+$ can be given by $(z_1, \dots, z_m) \to (|z_1|^2, \dots, |z_m|^2)$. If we restrict the above action to the standard poly-disk

$$(D^2)^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m \,:\, |z_i| \leq 1,\, i = 1, \dots, m\} \subset \mathbb{C}^m,$$

then the corresponding orbit space would be the standard cube $I^m \subset \mathbb{R}^m_+$.

Let $U_{\mathbb{R}}(K) \subset \mathbb{R}^m_+$ denote the orbit space $U(K)/T^m$. Note that if we regard $\mathbb{R}^m_+$ as a subset in $\mathbb{C}^m$, then $U_{\mathbb{R}}(K)$ is the "real part": $U_{\mathbb{R}}(K) = U(K) \cap \mathbb{R}^m_+$.

Definition 2.11. The *equivariant moment-angle complex* $\mathcal{Z}_K \subset \mathbb{C}^m$ corresponding to a simplicial complex K is the T^m-space defined from the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ \downarrow & & \downarrow \\ \mathcal{C}_K & \longrightarrow & I^m, \end{array}$$

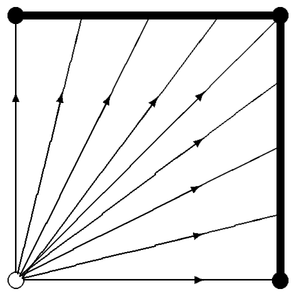


FIGURE 2. The retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ for $K = \partial\Delta^{m-1}$.

where the right vertical arrow denotes the orbit map for the diagonal action of T^m, and the lower horizontal arrow denotes the embedding of the cubical complex $\mathcal{C}_K$ to I^m.

Lemma 2.12. *$\mathcal{C}_K \subset U_{\mathbb{R}}(K)$ and $\mathcal{Z}_K \subset U(K)$.*

Proof. Definition 2.11 shows that the second assertion follows from the first one. To prove the first assertion we mention that if a point $a = (y_1, \ldots, y_m) \in \mathcal{C}_K$ has $y_{i_1} = \ldots = y_{i_k} = 0$, then $v_I = \{v_{i_1}, \ldots, v_{i_k}\}$ is a simplex of K, hence $L_I \not\subset \mathcal{A}(K)$. □

Lemma 2.13. *$U(K)$ is equivariantly homotopy equivalent to the moment-angle complex $\mathcal{Z}_K$.*

Proof. We construct a retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ that is covered by an equivariant retraction $U(K) \to \mathcal{Z}_K$. The latter would be a required homotopy equivalence.

The retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ is constructed inductively. We start from the boundary complex of an $(m-1)$-simplex and remove simplices of positive dimensions until we obtain K. On each step we construct a retraction, and the composite map would be required retraction r. If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then $U_{\mathbb{R}}(K) = \mathbb{R}^m_+ \setminus \{0\}$ and the retraction r is shown on Figure 2. Now suppose that the simplicial complex K is obtained by removing one $(k-1)$-dimensional simplex $v_J = \{v_{j_1}, \ldots, v_{j_k}\}$ from simplicial complex K'. By the inductive hypothesis, the lemma holds for K', that is, there is a retraction $r' : U_{\mathbb{R}}(K') \to \mathcal{C}_{K'}$ with the required properties. Let us consider the face $C_J \subset I^m$ (see Definition 2.9). Since v_J is not a simplex of K, the point a having coordinates $y_{j_1} = \ldots = y_{j_k} = 0$, $y_i = 1$, $i \notin \{j_1, \ldots, j_k\}$, do not belong to $U(K)$. Hence, we may apply the retraction from Figure 2 on the face C_J, starting from the point a. Denote this retraction by r_J. Now take $r = r_J \circ r'$. It is easy to see that this r is exactly the required retraction. □

Example 2.14. 1) If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then $\mathcal{Z}_K$ is homeomorphic to $(2m-1)$-dimensional sphere S^{2m-1}.

2) If K is the dual to the boundary complex of a n-dimensional simple polytope P^n, then $\mathcal{Z}_K$ is homeomorphic to a smooth $(m+n)$-dimensional manifold. This manifold, denoted $\mathcal{Z}_P$, is the main object of study in [BP2].

Corollary 2.15. *The Borel construction $ET^m \times_{T^m} \mathcal{Z}_K$ is homotopy equivalent to B_TK.*

Proof. The retraction $r : U(K) \to \mathcal{Z}_K$ constructed in the proof of Lemma 2.13 is equivariant with respect to the T^m-actions on $U(K)$ and $\mathcal{Z}_K$. Since $B_TK = ET^m \times_{T^m} U(K)$, the corollary follows. □

In what follows we do not distinguish the Borel constructions $ET^m \times_{T^m} \mathcal{Z}_K$ and $B_TK = ET^m \times_{T^m} U(K)$.

Theorem 2.16. *The cellular embedding $i : \widetilde{B_TK} \hookrightarrow BT^m$ (see* Definition 2.6*) and the fibration $p : B_TK \to BT^m$ (see* (1)*) are homotopically equivalent. In particular, $\widetilde{B_TK}$ and B_TK are of same homotopy type.*

Proof. Let $\pi : \mathcal{Z}_K \to \mathcal{C}_K$ denote the orbit map for the torus action on the moment-angle complex $\mathcal{Z}_K$ (see Definition 2.11). For each subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ denote by B_I the following subset of the poly-disk $(D^2)^m$: $B_I = B_1 \times \cdots \times B_m \subset D^2 \times \cdots \times D^2 = (D^2)^m$, where B_i is the disk D^2 if $i \in I$, and B_i is the boundary S^1 of D^2 if $i \notin I$. Thus, $B_I \cong (D^2)^k \times T^{m-k}$, where $k = |I|$. It is easy to see that if C_I is a face of cubical complex $\mathcal{C}_K$ (see Definition 2.9) then $\pi^{-1}(C_I) = B_I$. Since for $I \subset J$ the B_I is canonically identified with a subset of B_J, we see that those B_I for which v_I is a simplex of K fit together to yield $\mathcal{Z}_K$. (This idea can be used to prove that $\mathcal{Z}_K$ is a smooth manifold provided that K is the dual to the boundary complex of a simple polytope, see [BP2, Theorem 2.4].)

For any simplex $v_I \subset K$ the subset $B_I \subset \mathcal{Z}_K$ is invariant with respect to the T^m-action on $\mathcal{Z}_K$. Hence, the Borel construction $B_TK = ET^m \times_{T^m} \mathcal{Z}_K$ is patched from Borel constructions $ET^m \times_{T^m} B_I$ (compare this with the local construction of B_TP from [DJ, p. 435]). The latter can be factorized as $ET^m \times_{T^m} B_I = \big(ET^k \times_{T^k} (D^2)^k\big) \times ET^{m-n}$, which is homotopically equivalent to BT_I^k. Hence, the restriction of the projection $p : B_TK \to BT^m$ to $ET^m \times_{T^m} B_I$ is homotopically equivalent to the embedding $BT_I^k \hookrightarrow BT^m$. These homotopy equivalences for all simplices $v_I \subset K$ fit together to yield a required homotopy equivalence between $p : B_TK \to BT^m$ and $i : \widetilde{B_TK} \hookrightarrow BT^m$. □

Corollary 2.17. *The complement $U(K)$ of a coordinate subspace arrangement is a homotopy fibre of the cellular embedding $i : \widetilde{B_TK} \hookrightarrow BT^m$.* □

Corollary 2.18. *The T^m-equivariant cohomology ring $H^*_{T^m}\big(U(K)\big)$ is isomorphic to the face ring $\mathbf{k}(K)$.*

Proof. We have $H^*_{T^m}\big(U(K)\big) = H^*\big(ET^m \times_{T^m} U(K)\big) = H^*(B_TK)$. Now, the corollary follows from Lemma 2.8 and Theorem 2.16. □

3. Cohomology ring of $U(K)$

Suppose that we are given a $\mathbf{k}[v_1,\dots,v_m]$-free resolution of the face ring $\mathbf{k}(K)$ as a graded module over the polynomial ring $\mathbf{k}[v_1,\dots,v_m]$:

$$0 \to R^{-h} \xrightarrow{d^{-h}} R^{-h+1} \xrightarrow{d^{-h+1}} \cdots \to R^{-1} \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} \mathbf{k}(K) \to 0 \tag{2}$$

(note that the Hilbert syzygy theorem shows that $h \le m$ above). Applying the functor $\otimes_{\mathbf{k}[v_1,\dots,v_m]}\mathbf{k}$ to (2) we obtain a cochain complex:

$$0 \longrightarrow R^{-h} \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow \cdots \longrightarrow R^0 \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow 0,$$

whose cohomology modules are denoted $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$. Since all R^{-i} in (2) are graded $\mathbf{k}[v_1,\dots,v_m]$-modules, $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big) = \bigoplus_j \operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ is a graded $\mathbf{k}$-module, and

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big) = \bigoplus_{i,j} \operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big) \tag{3}$$

becomes a *bigraded* $\mathbf{k}$-module. Note that its non-zero elements have non-positive first grading and non-negative even second grading (since $\deg v_i{=}2$). The bigraded $\mathbf{k}$-module (3) can be also regarded as a one-graded module with respect to the total degree $-i+j$. The Betti numbers

$$\beta^{-i}\big(\mathbf{k}(K)\big) = \dim_{\mathbf{k}} \operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

and

$$\beta^{-i,2j}\big(\mathbf{k}(K)\big) = \dim_{\mathbf{k}} \operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

are of great interest in geometric combinatorics; they were studied by different authors (see, for example, [St]). We mention only one theorem due to Hochster, which reduces calculation of $\beta^{-i,2j}\big(\mathbf{k}(K)\big)$ to calculating the homology of subcomplexes of K.

Theorem 3.1 (Hochster [Ho], [St]). *The Hilbert series $\sum_j \beta^{-i,2j}\big(\mathbf{k}(K)\big)t^{2j}$ of $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ can be calculated as*

$$\sum_j \beta^{-i,2j}\big(\mathbf{k}(K)\big)t^{2j} = \sum_{I\subset\{v_1,\dots,v_m\}} \big(\dim_{\mathbf{k}} \tilde H_{|I|-i-1}(K_I)\big)t^{2|I|},$$

where K_I is the subcomplex of K consisting of all simplices with vertices in I. □

Note that calculation of $\beta^{-i,2j}\big(\mathbf{k}(K)\big)$ using this theorem is very involved even for small K.

It turns out that $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ is a bigraded *algebra* in a natural way, and the associated one-graded algebra is exactly $H^*\big(U(K)\big)$:

Theorem 3.2. *The following isomorphism of graded algebras holds:*

$$H^*\big(U(K)\big) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} \widetilde{U(K)} & \longrightarrow & ET^m \\ \big\downarrow & & \big\downarrow \\ \widetilde{B_TK} & \xrightarrow{i} & BT^m, \end{array} \tag{4}$$

where the left vertical arrow is the induced fibre bundle. Corollary 2.17 shows that $\widetilde{U(K)}$ is homotopically equivalent to $U(K)$.

From (4) we obtain that the cellular cochain algebras $C^*(\widetilde{B_TK})$ and $C^*(ET^m)$ are modules over $C^*(BT^m)$. It is clear from the proof of Lemma 2.8 that $C^*(\widetilde{B_TK}) = \mathbf{k}(K)$ and $i^* : C^*(BT^m) = \mathbf{k}[v_1, \dots, v_m] \to \mathbf{k}(K) = C^*(\widetilde{B_TK})$ is the quotient epimorphism. Since ET^m is contractible, we have a chain equivalence $C^*(ET^m) \to \mathbf{k}$. Therefore, there is an isomorphism

$$\operatorname{Tor}_{C^*(BT^m)}\bigl(C^*(\widetilde{B_TK}), C^*(ET^m)\bigr) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K), \mathbf{k}\bigr). \tag{5}$$

The Eilenberg–Moore spectral sequence (see [Sm, Theorem 1.2]) of commutative square (4) has the E_2-term

$$E_2 = \operatorname{Tor}_{H^*(BT^m)}\bigl(H^*(\widetilde{B_TK}), H^*(ET^m)\bigr)$$

and converges to $\operatorname{Tor}_{C^*(BT^m)}(C^*(\widetilde{B_TK}), C^*(ET^m))$. Since

$$\operatorname{Tor}_{H^*(BT^m)}(H^*(\widetilde{B_TK}), H^*(ET^m)) = \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K), \mathbf{k}\bigr),$$

it follows from (5) that the spectral sequence collapses at the E_2 term, that is, $E_2 = E_\infty$. Now, Proposition 3.2 of [Sm] shows that the module $\operatorname{Tor}_{C^*(BT^m)}\bigl(C^*(\widetilde{B_TK}), C^*(ET^m)\bigr)$ is an algebra isomorphic to $H^*\bigl(\widetilde{U(K)}\bigr)$, which concludes the proof. □

Our next theorem gives an explicit description of the algebra $H^*\bigl(U(K)\bigr)$ as the cohomology algebra of a simple differential bigraded algebra. We consider the tensor product $\mathbf{k}(K) \otimes \Lambda[u_1, \dots, u_m]$ of the face ring $\mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]/I$ and an exterior algebra $\Lambda[u_1, \dots, u_m]$ on m generators and make it a differential bigraded algebra by setting

$$\begin{aligned} &\operatorname{bideg} v_i = (0, 2), \quad \operatorname{bideg} u_i = (-1, 2), \\ &d(1 \otimes u_i) = v_i \otimes 1, \quad d(v_i \otimes 1) = 0, \end{aligned} \tag{6}$$

and requiring that d be a derivation of algebras.

Theorem 3.3. *The following isomorphism of graded algebras holds:*

$$H^*\bigl(U(K)\bigr) \cong H\bigl[\mathbf{k}(K) \otimes \Lambda[u_1, \dots, u_m], d\bigr],$$

where in the right hand side stands the one-graded algebra associated to the bigraded cohomology algebra.

Proof. One can make $\mathbf{k}$ a $\mathbf{k}[v_1,\ldots,v_m]$-module by means of the homomorphism that sends 1 to 1 and v_i to 0. Let us consider the Koszul resolution (see, for example [Ma, Chapter VII, § 2]) of $\mathbf{k}$ regarded as a $\mathbf{k}[v_1,\ldots,v_m]$-module:

$$\big[\mathbf{k}[v_1,\ldots,v_m]\otimes\Lambda[u_1,\ldots,u_m],d\big],$$

where the differential d is defined as in (6). Since the bigraded torsion product $\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\ ,\ )$ is a symmetric function of its arguments, one has

$$\mathrm{Tor}_\Gamma\big(\mathbf{k}(K),\mathbf{k}\big)=H\big[\mathbf{k}(K)\otimes_\Gamma\Gamma\otimes\Lambda[u_1,\ldots,u_n],d\big]=\big[\Gamma\otimes\Lambda[u_1,\ldots,u_m],d\big],$$

where we denoted $\Gamma=\mathbf{k}[v_1,\ldots,v_m]$. Since $H^*\big(U(K)\big)\cong\mathrm{Tor}_\Gamma\big(\mathbf{k}(K),\mathbf{k}\big)$ by Theorem 3.2, we obtain the required isomorphism □

Note that the above theorem not only calculates the cohomology algebra of $U(K)$, but also makes this algebra *bigraded.*

Corollary 3.4. *The Leray–Serre spectral sequence of the bundle $\widetilde{U(K)}\to\widetilde{B_TK}$ with fibre T^m (see (4)) collapses at the E_3 term.*

Proof. The spectral sequence under consideration converges to $H^*\big(\widetilde{U(K)}\big)=H^*\big(U(K)\big)$ and has

$$E_2=H^*(\widetilde{B_TK})\otimes H^*(T^m)=\mathbf{k}(K)\otimes\Lambda[u_1,\ldots,u_m].$$

It is easy to see that the differential in the E_2 term acts as in (6). Hence, $E_3=H[E_2,d]=H\big[\mathbf{k}(K)\otimes\Lambda[u_1,\ldots,u_m]\big]=H^*\big(U(K)\big)$ by Theorem 3.3. □

Proposition 3.5. *Suppose that a monomial*

$$v_{i_1}^{\alpha_1}\ldots v_{i_p}^{\alpha_p}u_{j_1}\ldots u_{j_q}\in\mathbf{k}(K)\otimes\Lambda[u_1,\ldots,u_m],$$

where $i_1<\ldots<i_p$, $j_1<\ldots<i_q$, represents a non-trivial cohomology class in $H^*\big(U(K)\big)$. Then $\alpha_1=\ldots=\alpha_p=1$, $\{v_{i_1},\ldots,v_{i_p}\}$ spans a simplex of K, and $\{i_1,\ldots,i_p\}\cap\{j_1,\ldots,j_q\}=\varnothing$.

Proof. See [BP2, Lemma 5.3]. □

As it was mentioned above (see Example 2.14), if K is the boundary complex of a convex simplicial polytope (or, equivalently, K is the dual to the boundary complex of a simple polytope) or at least a simplicial sphere, then $U(K)$ has homotopy type of a smooth manifold $\mathcal{Z}_K$. It was shown in [BP2, Theorem 2.10] that the corresponding homotopy equivalence can be interpreted as the orbit map $U(K)\to U(K)/\mathbb{R}^{m-n}\cong\mathcal{Z}_K$ with respect to a certain action of $\mathbb{R}^{m-n}$ on $U(K)$.

The coordinate subspace arrangement $\mathcal{A}(K)$ and its complement $U(K)$ play important role in the theory of toric varieties and symplectic geometry (see, for example, [Au], [Ba], [Co]). More precisely, any n-dimensional simplicial toric variety M defined by a (simplicial) fan Σ in $\mathbb{Z}^n$ with m one-dimensional cones can be obtained as the geometric quotient $U(K_\Sigma)/G$. Here G is a subgroup of the complex torus $(\mathbb{C}^*)^m$ isomorphic to $(\mathbb{C}^*)^{m-n}$ and K_Σ

is the simplicial complex defined by the fan Σ (i-simplices of K_Σ correspond to $(i+1)$-dimensional cones of Σ). A smooth projective toric variety M is a symplectic manifold of real dimension $2n$. This manifold can be constructed by the process of *symplectic reduction* in the following way. Let $G_{\mathbb{R}} \cong T^{m-n}$ denote the maximal compact subgroup of G, and let $\mu : \mathbb{C}^m \to \mathbb{R}^{m-n}$ be the moment map for the Hamiltonian action of $G_{\mathbb{R}}$ on $\mathbb{C}^m$. Then for each regular value $a \in \mathbb{R}^{m-n}$ of μ there is a diffeomorphism

$$\mu^{-1}(a)/G_{\mathbb{R}} \longrightarrow U(K_\Sigma)/G = M$$

(see [Co] for more information). In this situation it can be easily seen that $\mu^{-1}(a)$ is exactly our manifold $\mathcal{Z}_K$ for $K = K_\Sigma$.

Example 3.6. Let $G \cong \mathbb{C}^*$ be the diagonal subgroup in $(\mathbb{C}^*)^{n+1}$ and K_Σ be the boundary complex of an n-simplex. Then $U(K_\Sigma) = \mathbb{C}^{n+1} \setminus \{0\}$ and $M = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$ is the complex projective space $\mathbb{C}P^n$. The moment map $\mu : \mathbb{C}^m \to \mathbb{R}$ takes $(z_1, \ldots, z_m) \in \mathbb{C}^m$ to $\frac{1}{2}(|z_1|^2 + \ldots + |z_m|^2)$ and for $a \neq 0$ one has $\mu^{-1}(a) \cong S^{2n+1} \cong \mathcal{Z}_K$ (see Example 2.14).

In the case when K is a simplicial sphere (hence, the complement $U(K)$ is homotopically equivalent to the smooth manifold $\mathcal{Z}_K$), there is Poincaré duality defined in the cohomology ring of $U(K)$.

Proposition 3.7. *Suppose that K is a simplicial sphere of dimension $n-1$, hence, $U(K)$ is homotopically equivalent to the smooth manifold $\mathcal{Z}_K$. Then*

1) The Poincaré duality in $H^*\big(U(K)\big)$ regards the bigraded structure defined by Theorem 3.3. *More precisely, if $\alpha \in H^{-i,2j}\big(U(K)\big)$ is a cohomology class, then its Poincaré dual $D\alpha$ belongs to $H^{-(m-n)+i,2(m-j)}$.*

2) Let $\{v_{i_1}, \ldots, v_{i_n}\}$ be an $(n-1)$-simplex of K and let $j_1 < \ldots < j_{m-n}$, $\{i_1, \ldots, i_n, j_1, \ldots, j_{m-n}\} = \{1, \ldots, m\}$. Then the value of the element

$$v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} \in H^{m+n}\big(U(K)\big) \cong H^{m+n}(\mathcal{Z}_K)$$

on the fundamental class of $\mathcal{Z}_K$ equals ± 1.

3) Let $\{v_{i_1}, \ldots, v_{i_n}\}$ and $\{v_{i_1}, \ldots, v_{i_{n-1}}, v_{j_1}\}$ be two $(n-1)$-simplices of K having common $(n-2)$-face $\{v_{i_1}, \ldots, v_{i_{n-1}}\}$, and $j_1, \ldots, j_{m-n}$ be as in 2). Then

$$v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} = v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}}$$

in $H^{m+n}\big(U(K)\big)$.

Proof. For the proof of 1) and 2) see [BP2, Lemma 5.1]. To prove 3) we just mention that

$$\begin{aligned} d(v_{i_1} &\cdots v_{i_{n-1}} u_{i_n} u_{j_1} u_{j_2} \cdots u_{j_{m-n}}) \\ &= v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} - v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}} \end{aligned}$$

in $\mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m]$ (see (6)). □

A simplicial complex K is called *Cohen–Macaulay*, if its face ring $\mathbf{k}(K)$ is a Cohen–Macaulay algebra, that is, $\mathbf{k}(K)$ is a finite-dimensional free module over a polynomial ring $\mathbf{k}[t_1,\dots,t_n]$ (here n is the maximal number of algebraically independent elements of $\mathbf{k}(K)$). Equivalently, $\mathbf{k}(K)$ is a Cohen–Macaulay algebra if it admits a *regular sequence* $\{\lambda_1,\dots,\lambda_n\}$, that is, a set of n homogeneous elements such that λ_{i+1} is not a zero divisor in $\mathbf{k}(K)/(\lambda_1,\dots,\lambda_i)$ for $i=0,\dots,n-1$. If K is a Cohen–Macaulay complex and $\mathbf{k}$ is of infinite characteristic, then $\mathbf{k}(K)$ admits a regular sequence of degree-two elements (remember that we set $\deg v_i=2$ in $\mathbf{k}(K)$), that is, $\lambda_i=\lambda_{i1}v_1+\lambda_{i2}v_2+\ldots+\lambda_{im}v_m$, $i=1,\dots,n$.

Theorem 3.8. *Suppose that K is a Cohen–Macaulay complex and $J=(\lambda_1,\dots,\lambda_n)$ is an ideal in $\mathbf{k}(K)$ generated by a regular sequence. Then the following isomorphism of bigraded algebras holds:*

$$H^*\bigl(U(K)\bigr)\cong H\bigl[\mathbf{k}(K)/J\otimes\Lambda[u_1,\dots,u_{m-n}],d\bigr],$$

where the gradings and differential in the right hand side are defined as follows:

$$\operatorname{bideg} v_i=(0,2),\quad \operatorname{bideg} u_i=(-1,2);$$
$$d(1\otimes u_i)=\lambda_i\otimes 1,\quad d(v_i\otimes 1)=0,$$

Hence, in the case when K is Cohen–Macaulay, the cohomology of $U(K)$ can be calculated via the finite-dimensional differential algebra $\mathbf{k}(K)/J\otimes\Lambda[u_1,\dots,u_{m-n}]$ instead of infinite-dimensional algebra $\mathbf{k}(K)\otimes\Lambda[u_1,\dots,u_m]$ from Theorem 3.3.

Example 3.9. Let K be the boundary complex of an $(m-1)$-dimensional simplex. Then $\mathbf{k}(K)=\mathbf{k}[v_1,\dots,v_m]/(v_1\cdots v_m)$. It easy to check that only non-trivial cohomology classes in $H\bigl[\mathbf{k}(K)\otimes\Lambda[u_1,\dots,u_m],d\bigr]$ (see Theorem 3.3) are represented by the cocycles 1 and $v_1v_2\cdots v_{m-1}u_m$ or their multiples. We have $\deg(v_1v_2\cdots v_{m-1}u_m)=2m-1$, and Proposition 3.7 shows that $v_1v_2\cdots v_{m-1}u_m$ is the fundamental cohomological class of $\mathcal{Z}_K\cong S^{2m-1}$ (see Example 2.14 1)).

Example 3.10. Let K be a disjoint union of m vertices. Then $U(K)$ is obtained by removing from $\mathbb{C}^m$ all codimension-two coordinate subspaces $z_i=z_j=0$, $i,j=1,\dots,m$ (see Example 2.3), and $\mathbf{k}(K)=\mathbf{k}[v_1,\dots,v_m]/I$, where I is the ideal generated by all monomials v_iv_j, $i\ne j$. It is easily deduced from Theorem 3.3 and Proposition 3.5 that any cohomology class of $H^*\bigl(U(K)\bigr)$ is represented by a linear combination of monomial cocycles $v_{i_1}u_{i_2}u_{i_3}\cdots u_{i_k}\subset\mathbf{k}(K)\otimes\Lambda[u_1,\dots,u_m]$ such that $k\ge 2$, $i_p\ne i_q$ for $p\ne q$. For each k there $m\binom{m-1}{k-1}$ such monomials, and there $\binom{m}{k}$ relations between them (each relation is obtained by calculating the differential of $u_{i_1}\cdots u_{i_k}$). Since $\deg(v_{i_1}u_{i_2}u_{i_3}\cdots u_{i_k})=k+1$, we have

$$\dim H^0\bigl(U(K)\bigr)=1,\quad H^1\bigl(U(K)\bigr)=H^2\bigl(U(K)\bigr)=0,$$
$$\dim H^{k+1}\bigl(U(K)\bigr)=m\tbinom{m-1}{k-1}-\tbinom{m}{k},\ 2\le k\le m,$$

and the multiplication in the cohomology is trivial.

In particular, for $m=3$ we have 6 three-dimensional cohomology classes v_iu_j, $i\ne j$, with 3 relations $v_iu_j=v_ju_i$, and 3 four-dimensional cohomology classes $v_1u_2u_3$, $v_2u_1u_3$, $v_3u_1u_2$ with one relation

$$v_1u_2u_3-v_2u_1u_3+v_3u_1u_2=0.$$

Hence, $\dim H^3\bigl(U(K)\bigr)=3$, $\dim H^4\bigl(U(K)\bigr)=2$, and the multiplication is trivial.

Example 3.11. Let K be a boundary complex of an m-gon ($m\ge 4$). Then, as it have been mentioned above, the moment-angle complex $\mathcal{Z}_K$ is a smooth manifold of dimension $m+2$, and $U(K)$ is homotopically equivalent to $\mathcal{Z}_K$. We have $\mathbf{k}(K)=\mathbf{k}[v_1,\dots,v_m]/I$, where I is generated by monomials v_iv_j such that $i\ne j\pm 1$. (Here we use the agreement $v_{m+i}=v_i$ and $v_{i-m}=v_i$.) The cohomology rings of these manifolds were calculated in [BP2]. We have

$$\dim H^k\bigl(U(K)\bigr)=\begin{cases}1 & \text{if } k=0 \text{ or } m+2;\\ 0 & \text{if } k=1,2,m \text{ or } m+1;\\ (m-2)\binom{m-2}{k-2}-\binom{m-2}{k-1}-\binom{m-2}{k-3} & \text{if } 3\le k\le m-1.\end{cases}$$

For example, in the case $m=5$ there 5 generators of $H^3\bigl(U(K)\bigr)$ represented by the cocycles $v_iu_{i+2}\in\mathbf{k}(K)\otimes\Lambda[u_1,\dots,u_5]$, $i=1,\dots,5$, and 5 generators of $H^4\bigl(U(K)\bigr)$ represented by the cocycles $v_ju_{j+2}u_{j+3}$, $j=1,\dots,5$. As it follows from Proposition 3.7, the product of cocycles v_iu_{i+2} and $v_ju_{j+2}u_{j+3}$ represents a non-trivial cohomology class in $H^7\bigl(U(K)\bigr)$ (the fundamental cohomology class up to sign) if and only if $\{i,i+2,j,j+2,j+3\}=\{1,2,3,4,5\}$. Hence, for each cohomology class $[v_iu_{i+2}]$ there is a unique (Poincaré dual) cohomology class $[v_ju_{j+2}u_{j+3}]$ such that the product $[v_iu_{i+2}]\cdot[v_ju_{j+2}u_{j+3}]$ is non-trivial.

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