Torus actions in topology and combinatorics

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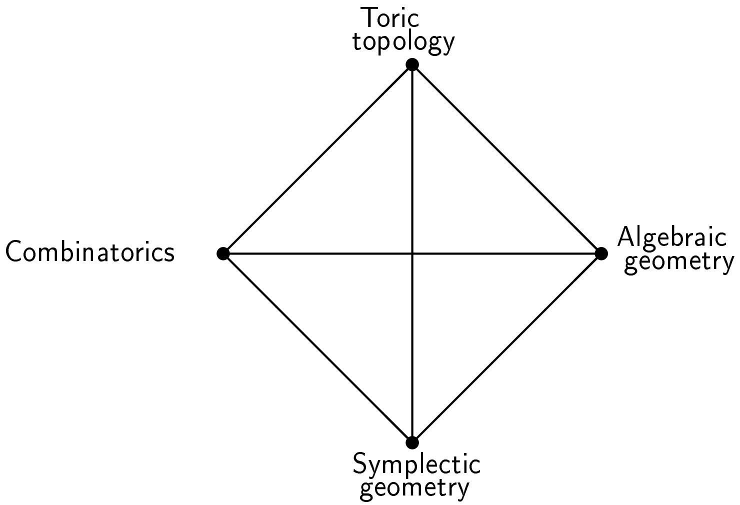
«Toric Tetrahedron»



1. Combinatorics.

$\mathcal{K}$ simplicial complex on the set $[m] = \{1, \ldots, m\}$
(A collection of subsets in $[m]$ closed under inclusion).

$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex (or face) of dimension $k-1$.
Always assume $\varnothing \in \mathcal{K}$.

$f_i = f_i(\mathcal{K})$ the number of faces (simplices) of dimension i.
Let $\dim \mathcal{K} = n-1$.

$\boldsymbol{f}(\mathcal{K}) = (f_0, f_1, \ldots, f_{n-1})$ the f-vector. $\quad f_0 = m$.
The h-vector $\boldsymbol{h}(\mathcal{K}) = (h_0, h_1 \ldots, h_n)$ is defined from the identity

$$h_0 t^n + h_1 t^{n-1} + \cdots + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}.$$

$\mathcal{K}$ is a triangulated sphere if $|\mathcal{K}| \cong S^{n-1}$.
Example: the boundary of a convex n-dimensional simplicial polytope.

Question: how to characterise the f-vectors (or h-vectors) for interesting classes of simplicial complexes (e.g., polytopes, triangulated spheres or triangulated manifolds)?

Examples of restrictions on $\boldsymbol{f}(\mathcal{K})$:

$f_0 - f_1 + \ldots + (-1)^{n-1} f_{n-1} = \chi(\mathcal{K}) \quad \Longleftrightarrow \quad h_n - h_0 = \chi(\mathcal{K}) - \chi(S^{n-1})$;

$2f_{n-2} = nf_{n-1}$ if $\mathcal{K}$ is a triangulated sphere or manifold;

$h_i = h_{n-i}$ if $|\mathcal{K}| \cong S^{n-1}$ (Dehn–Sommerville relations);

$f_1 \leqslant \binom{f_0}{2}$;

$f_0 \geqslant n+1 \quad \Longleftrightarrow \quad h_0 \leqslant h_1$ if $|\mathcal{K}| \cong S^{n-1}$;

$f_1 \geqslant nf_0 - \binom{n+1}{2} \quad \Longleftrightarrow \quad h_1 \leqslant h_2$ if $|\mathcal{K}| \cong S^{n-1}$.

Theorem (Billera–Lee, Stanley, 1980)

The following conditions are necessary and sufficient for a collection $(f_0, f_1, \ldots, f_{n-1})$ to be the f-vector of a simplicial polytope:

(a) $h_i = h_{n-i}$ *for* $i = 0, \ldots, n$;

(b) $h_0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_{[n/2]}$;

(c) … (a restriction on the growth of h_i*).*

In the proof, a projective toric variety X_P is assigned to a polytope P. The cohomology of X_P satisfies

$$\dim H^{2i}(X_P, \mathbb{Q}) = h_i(P).$$

Then (a) follows from Poincaré duality, while (b) and (c) follow from the Hard Lefschetz Theorem for projective varieties.

Problem (McMullen's conjecture)

Is it true that the same conditions (a)–(b) characterise the f-vectors of triangulated spheres?

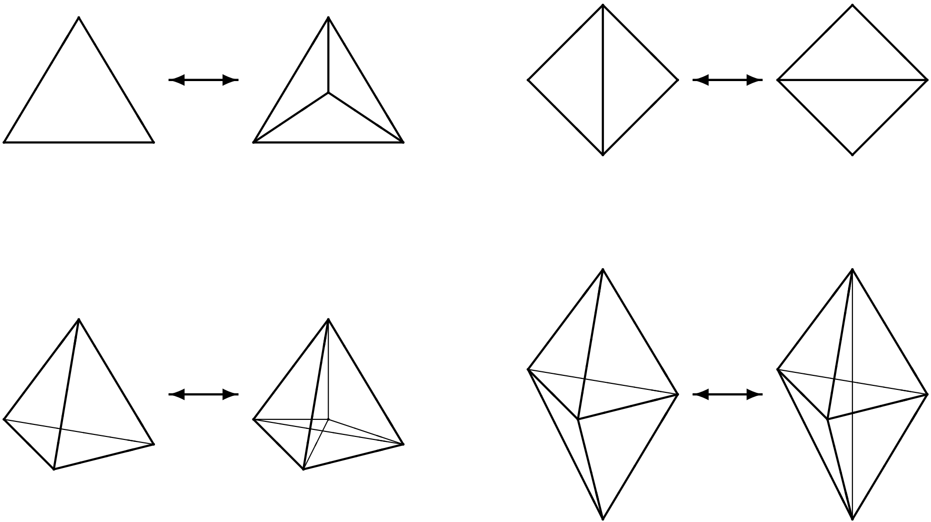
How to obtain triangulations which do not arise from polytopes?

There are classical examples:

- Barnette sphere and Brückner sphere in dimension 3;
- Double suspension on the Poncaré sphere in dimension 5.

More examples can be constructed using the operations of suspension, join, and bistellar moves (or flips).

Bistellar moves in dimension 2 and 3



Theorem (Pachner)

Two triangulated manifolds are piecewise linearly equivalent if and only if one can be taken into another by a sequence of bistellar moves.

It follows that if we start with a non-piecewise-linear sphere triangulation (e.g., from the double suspension of the Poincaré sphere) and apply bistellar moves to it, then we never end up in a polytopal triangulation.

The behaviour of the h-vector under bistellar moves is easily controlled. There are software packages (*Bistellar*) executing this procedure.

However, no counterexamples to McMullen's conjecture have been found yet on this way...

2. Topology.

Consider a unit polydisc in $\mathbb{C}^m$:

$$\mathbb{D}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1\}.$$

For each $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$B_I := \{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\}.$$

Define the moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is an invariant subset with respect to the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| = 1, \quad i = 1, \ldots, m\}.$$

Given a pair of subsets $W \subset X$ and $I \subset [m]$, set

$$(X, W)^I = \{(x_1, \dots, x_m) \in X^m : x_j \in W \text{ for } j \notin I\} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,$$

and define the polyhedral product of the pair (X, W) by

$$(X, W)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (X, W)^I \subset X^m.$$

Then $\mathcal{Z}_{\mathcal{K}} = (\mathbb{D}, \mathbb{S})^{\mathcal{K}}$, where $\mathbb{S}$ is the unit circle.
Another example: the complement of a coordinate subspace arrangement

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\}$$
$$= (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times\Big),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Clearly, $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$.

Theorem (Buchstaber–Panov)

(a) There is a deformation retraction $U(\mathcal{K}) \to \mathcal{Z}_{\mathcal{K}}$*;*
(b) Let $|\mathcal{K}| \cong S^{n-1}$. *Then* $\mathcal{Z}_K$ *is manifold.*

3. Combinatorial commutative algebra.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$.
The face ring (or the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} : \{i_1, \ldots, i_k\} \notin \mathcal{K}\big), \qquad \deg v_i = 2.$$

Theorem (Buchstaber–Panov)

There is an isomorphism of (bi)graded algebras

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \operatorname{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}\big(\mathbb{Z}[\mathcal{K}], \mathbb{Z}\big).$$

Corollary

If $\mathcal{K}$ is a triangulation of an $(n-1)$-dimensional manifolds, then Poincaré duality for $\mathcal{Z}_\mathcal{K}$ implies the relations

$$h_{n-i} - h_i = (-1)^i \big(\chi(\mathcal{K}) - \chi(S^{n-1})\big) \binom{n}{i}$$

(the generalised Dehn–Sommerville relations).

The dimensions of the bigraded components of the Tor-groups,

$$\beta^{-i,2j}(\mathbb{Z}[\mathcal{K}]) := \dim \operatorname{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z})$$

are subtle combinatorial invariants of $\mathcal{K}$.

The face numbers of $\mathcal{K}$ are expressed in terms of $\beta^{-i,2j}(\mathbb{Z}[\mathcal{K}])$.

The bigraded Betti numbers $\beta^{-i,2j}(\mathbb{Z}[\mathcal{K}])$ can be also computed via the simplicial cohomology of $\mathcal{K}$:

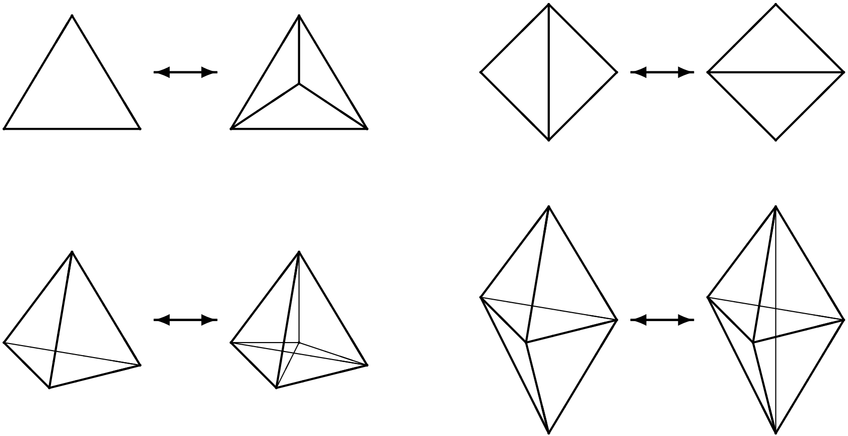
Theorem (Hochster)

$$\operatorname{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) \cong \bigoplus_{J\subset[m],\ |J|=j} \widetilde{H}^{j-i-1}(K|_J),$$

where $K|_J$ is the full subcomplex (the restriction of $\mathcal{K}$ to $J \subset \{1,\dots,m\}$).

The numbers $\beta^{-i,2j}(\mathbb{Z}[\mathcal{K}])$ can be also computed effectively using the commutative algebra software package *Macaulay 2*.

4. Bistellar moves in topology and geometry



Theorem (Pachner, 1987)

For any two PL homeomorphic triangulations of the same manifold, the first one can be transformed to the second one by a finite sequence of bistellar moves and simplicial isomorphisms.

Applications of bistellar moves, stellar subdivisions, and related operations

1. Construction of a local combinatorial formula for the first rational Pontryagin class of a triangulated manifold (Gaifullin, 2004).
2. The well-known Gal conjecture on the properties of the face numbers of flag simple polytopes is proved for the case of nestohedra (Volodin, 2010). Precise lower and upper bounds for the flag numbers are proved for many important families of flag simple polytopes, which appear in different areas of mathematics. (Buchstaber–Volodin, 2011). These estimates are based on the fact that all polytopes in these families can be obtained from a cube by consecutive truncations of codimension 2 faces.
3. The formulae for the numbers of multiplicative generators in a given dimension of the rings of flag vectors of convex polytopes (Buchstaber-Erokhovets, 2011).

Applications of bistellar moves, stellar subdivisions, and related operations

4. Existence of a formula for the volume of a simplicial 4-dimensional polyhedron from its combinatorial structure and the set of its edge lengths. The volume of an arbitrary flexible 4-dimensional polyhedron is constant. (Gaifullin, 2011)

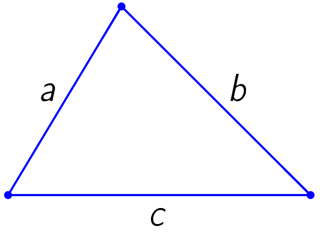
Combinatorial computation of the Pontryagin classes

- Pontryagin classes are classical invariants of manifolds. Their definition uses a smooth structure on the manifold.
- In 1957/58 Rokhlin-Shvarts and, independently, Thom proved the invariance of the rational Pontryagin classes under PL homeomorphisms. This result leads naturally to the problem of combinatorial computation of the rational Pontryagin classes of a manifold from a triangulation of it.
- Important results on this problem have been obtained by Gabrielov–Gelfand–Losik (1975), MacPherson (1977), Levitt-Rourke (1978), Cheeger (1983), Gelfand–MacPherson (1992). However, a complete solution has not been achieved.

Theorem (Gaifullin, 2004)

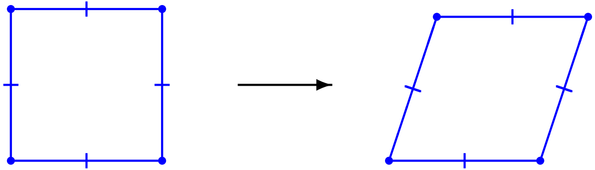
1. *There exists an explicit algorythm that for any (oriented) m-dimensional combinatorial manifold K, computes an $(m-4)$-dimensional simplicial cycle $Z \in C_{m-4}(K, \mathbb{Q})$ representing the Poincaré dual of the first Pontryagin class of K.*
2. *The cycle Z is computed from the triangulation K locally. This means that*
$$Z = \sum_{\dim \sigma = m-4} c_\sigma \sigma,$$
where the coefficient c_σ depends only on the combinatorial structure of the triangulation K in a neighbourhood of a simplex σ. To be more precise, c_σ depends only on the combinatorial structure of the link L_σ of σ in K. (L_σ is a triangulation of a 3-sphere.)
3. *To compute the coefficient c_σ one needs to transform L_σ to the boundary of a 4-simplex by means of bistellar moves:*
$$L_\sigma = L_1 \rightsquigarrow L_2 \rightsquigarrow \cdots \rightsquigarrow L_k = \partial \Delta^4$$
and then to take the sum of contributions of all these bistellar moves.

Heron's formula



$$S^2 = p(p-a)(p-b)(p-c)$$

$$p = \frac{a+b+c}{2}$$



No formula for the area from edge lengths

Cayley–Menger formula

Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional simplex with vertices $p_0, p_1, \dots, p_n$ and let ℓ_{ij} be the length of the edge $p_i p_j$.

The Cayley–Menger determinant is given by

$$CM(p_0, \dots, p_n) = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \ell_{01}^2 & \ell_{02}^2 & \cdots & \ell_{0n}^2 \\ 1 & \ell_{01}^2 & 0 & \ell_{12}^2 & \cdots & \ell_{1n}^2 \\ 1 & \ell_{02}^2 & \ell_{12}^2 & 0 & \cdots & \ell_{2n}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_{0n}^2 & \ell_{1n}^2 & \ell_{2n}^2 & \cdots & 0 \end{vmatrix}$$

Then

$$V^2(\Delta) = \frac{(-1)^{n+1}}{2^n (n!)^2} CM(p_0, \dots, p_n).$$

Computation of the volume of a simplicial polyhedron

Problem

Suppose $n \geqslant 3$. For a given simplicial n-dimensional polyhedron in $\mathbb{R}^n$, can we find a formula for its volume in terms of its edge lengths?

Equivalent problem

Suppose $n \geqslant 3$. For a given n-dimensional polyhedron in $\mathbb{R}^n$, can we find a formula for its volume in terms of the intrinsic metrics of the faces, that is, in terms of the lengths of edges and diagonals of faces?

"There is a formula" means that the volume is a root of a polynomial

$$V^N + a_1(\ell)V^{N-1} + \cdots + a_N(\ell) = 0,$$

where a_j are polynomials in the edge lengths of the polyhedron.

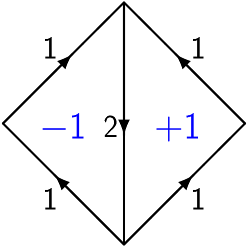
- $n = 3$: YES, Sabitov, 1996.
- $n = 4$: YES, Gaifullin, 2011.
- $n \geqslant 5$: UNKNOWN

What is a simplicial polyhedron in $\mathbb{R}^4$?

Naive answer: A region bounded by a closed 3-dimensional triangulated polyhedral surface.

More general answer: The boundary of a polyhedron is a 3-dimensional cycle in $\mathbb{R}^4$, i. e., a formal linear combination (with integral coefficients) of oriented convex 3-simplices in $\mathbb{R}^4$ such that its algebraic boundary is zero.

For any such manifold the generalised volume can be defined.



Main ideas of the proof ($n = 4$)

Theorem (Gaifullin, 2011)

For each combinatorial type of polyhedra there exists a polynomial relation

$$Q(V,\ell) = V^N + a_1(\ell)V^{N-1} + a_2(\ell)V^{N-2} + \ldots + a_N(\ell)$$

between the volume V of a polyhedron and the set ℓ of edge lengths of a polyhedron. Here $a_j(\ell)$ are polynomials in edge lengths with rational coefficients.

The proof of this theorem by induction on the number of vertices, then on the smallest vertex degree, etc. To simplify a polyhedron the following moves are used:

We can add (subtract) the boundary of a 4-dimensional convex simplex in $\mathbb{R}^4$ to the boundary of the polyhedron.

These moves are natural analogues of bistellar moves.

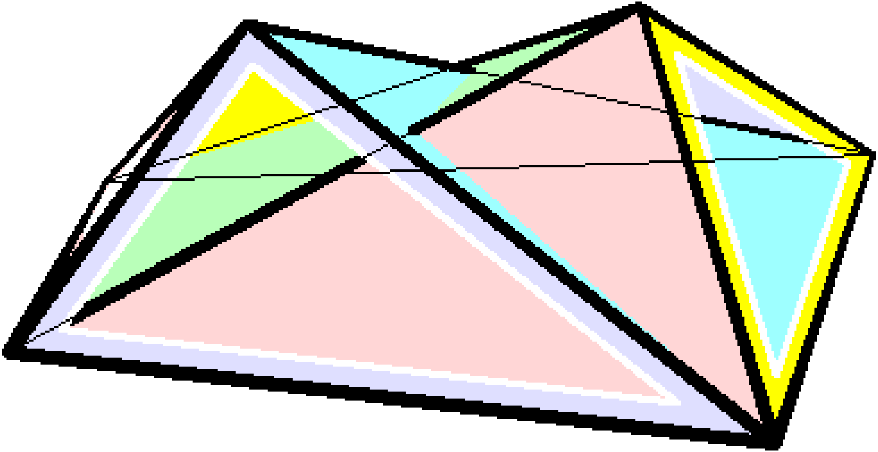
Flexible polyhedra

Definition

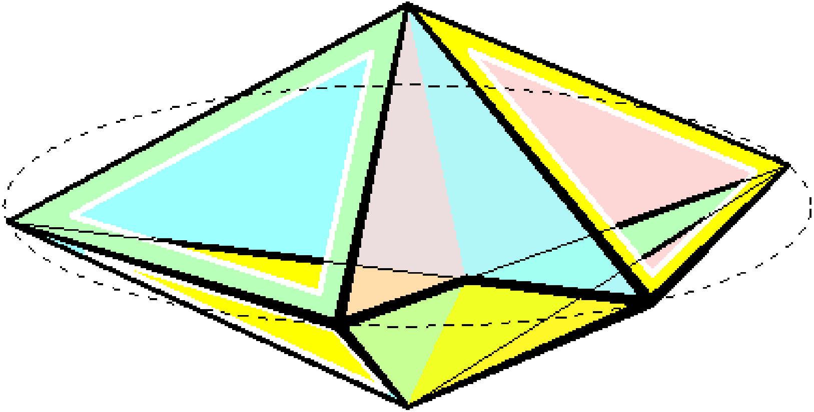
A *flex* of a polyhedron P is a continuous family of polyhedra P_t, $0 \leqslant t \leqslant 1$, of the same combinatorial type such that $P_0 = P$, the edge lengths of the polyhedra P_t are constant, and the polyhedra P_{t_1} and P_{t_2} are not congruent unless $t_1 = t_2$.

- The Cauchy Theorem: No convex polyhedron is flexible.
- Bricard, 1897: Flexible self-intersected octahedra.
- Connelly, 1977: First example of a flexible embedded polyhedron.
- Steffen, 1978: The simplest known flexible embedded polyhedron.
- Fogelsanger, 1988: Polyhedra in general position are not flexible.
- Walz, 1998, Stachel, 2000: Flexible 4-dimensional cross-polytopes.
- It is unknown if flexible polyhedra exist in $\mathbb{R}^n$, $n \geqslant 5$.

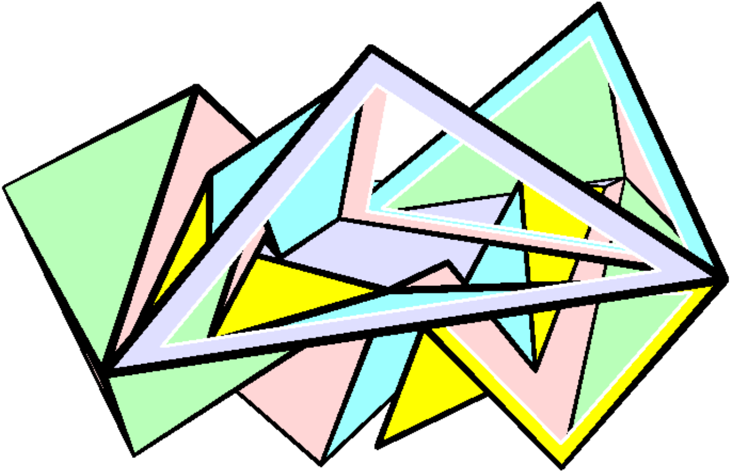
Bricard's flexible octahedron of the first type



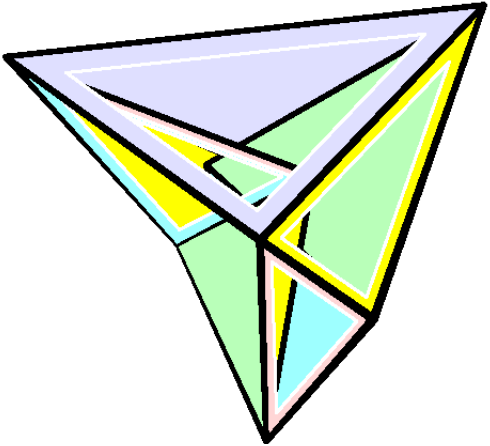
Bricard's flexible octahedron of the second type



Connelly's flexible polyhedron. The first example of an embedded flexible polyhedron



Steffen's flexible polyhedron. The simplest known embedded flexible polyhedron



The Bellows Conjecture

Conjecture (≈1978)

The generalized volume $V(P_t)$ of a flexible polyhedron is constant.

Theorem

1. (Sabitov, 1996) *The Bellows Conjecture is valid in dimension* 3.
2. (Gaifullin, 2011) *The Bellows Conjecture is valid in dimension* 4.

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